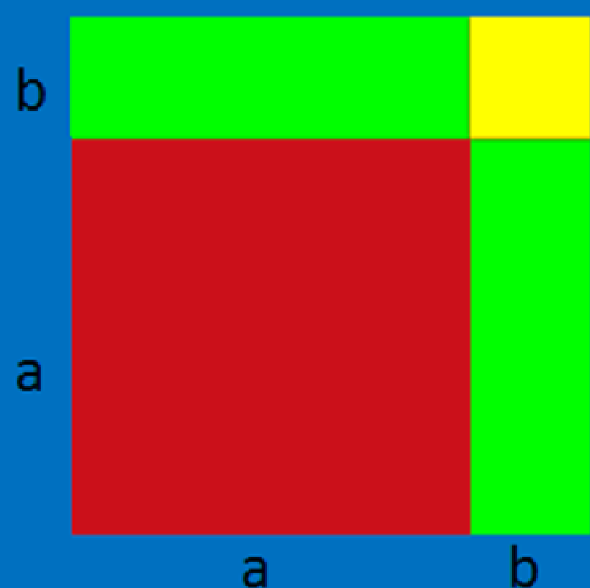
PRE-ALGEBRA

ALGEBRA 1



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ylani@outlook.com

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Introduction and Course Instructions

Hi, welcome to Algebra 1 . Here is how this course works:

First, take the Algebra 1 Pretest at the end of the Arithmetic Review e-book. Make sure you know how to work with numbers, decimals and fractions before you start on Algebra. In math, it never pays to skip over the basics because new skills build on your previous knowledge.

If you want to get a driver's license, you go to the Department of Motor Vehicles and show an examiner that you can drive a car properly. If you don't have the right skills, you'll be told to get more practice and come back later. It's the same for this course. In order to pass, you need to show that you can solve algebra problems properly. When you can do that, you have passed this course. Passing score is 85 – 100%, which is an A, so you don't have to worry about what grade you'll get. That’s the easy part. So what’s the hard part?

I'll just take the Quizzes so I can finish quickly

That strategy has already been tried but hasn't worked for anyone so far. If you just want to do the required work and get it over with, this course is probably not for you. Try a regular textbook instead. In this e-book material is provided so that you can learn from it, but you have to do that work. There are no long rows of almost identical problems because that’s boring. Instead there are web links that help you explore a topic. Take your time, and ask questions like “Why does that work?, “Is that really true?”, “How does it relate to things I know?”, “Could there be another way to do this?” Use a search engine and find more information about topics that even slightly interest you. Create nice portfolio pages to show off what you learned. Many of the web links offer practice problems, and you get to decide how many of them you should do. Do not rush ahead so that you can get the required 100% on the quiz as quickly as possible and finish the chapter. College admission tests don’t care how many chapters you finished. The only way to do well on them is by being good at algebra. The only way to get good at algebra is to read about algebra, think about algebra, and practice algebra until it gets easy. Suggested time for completing this course is one to two years.

Yeah, I'll do it Tomorrow

I attended an experimental public high school that had no fixed classes and no deadlines. At first it was great. I signed up for a lot of courses and got a good start on all of them. But soon there were just some things that were easier to do than others. I hated writing English essays, so I put those off till tomorrow. And tests, well maybe I wasn't quite ready, so they could be done tomorrow. It was a lot more fun to chat with my friends who were also putting some things off till tomorrow. Before I knew it, I was a long way from where I wanted to be. Eventually I realized that you can't stay in high school forever, and vague dreams of being rich and famous someday just aren't going to cut it. "Tomorrow" never comes, so you have to do as much as you can today.

We all look at famous athletes or musicians and think how lucky they are to have such natural talent. What we don't see is how hard these people practice every day to be good at what they do. Public school kids do math every day. I know it is hard to motivate yourself to do that if you are homeschooled, especially when you have to miss days and then make up for it the next day. Try to visit the course website every day you that you can. My personal strategy is "math first". I do math first every day, which has helped me get a lot more math done over time.

How to use the Chapter text

The text in this course was created to be readable like a regular book. It gives explanations rather than stepwise solutions, and hopefully it will help you see that you could use a different order of steps, or even completely different steps to arrive at the same solution. You may want to write your own steps on paper as you read. Follow along with the text and write out the equations and formulas as one item per line, so you can see the solution from top to bottom.

Also, math can never be read as quickly as ordinary text. Math writing is very condensed, and a complex idea can be written with only a few characters. You need to make sure that you really understand what you are reading. To help you see where to stop and check, the text has stop lines. Pause at the stop line and think about what you just read. Read the section again if you aren't sure. You may have to read the whole page, and then go back and read it again.

How to use the Internet

Although some web links are provided, they may be out of date. Fortunately the Internet has a lot of information on every subject, and most math sites provide practice problems. Use a search engine to find your own links to explore. Links mentioned in this e-book are often older non-profit websites that will show up in your browser as “not secure”. That just means that there wasn’t the money or time available to get a security certificate for the site. Since you will not enter any personal information or buy anything on these websites they are safe to use.

How to use the Quizzes

I made these quizzes for you, so you could check on what you have learned. Take the quiz after you have read the topic, thought about it, and done enough practice problems that you think you know it. If you prefer immediate feedback you can print the solutions, cover them with something, and check them one by one. Sometimes the answer in the solutions will differ from yours. If you don’t spot some obvious mistake you made, you should assume your answer is in fact correct, and try to prove it. After all, it is always possible that the answer key is wrong. This is where you actually learn how to do math.

How to Make Portfolio Pages:

In some districts homeschooled students are required to have portfolios. Some colleges will even look at them. Mostly however portfolios are a way to think about what you learned and express it in a personal way that you can be proud of. In math, you need to make sure that your portfolio is understandable. Ask yourself, "Will this still make sense if I look at it three years from now?" When you are reviewing for your college admission test, or needing to look something up when you are in college and taking a math course, you should be able to go back to your portfolio to help you remember what you learned in Algebra 1.

Portfolio Evaluation

Incomplete: The directions in the assignment have not been followed and/or question(s) asked in the assignment have not been answered.

Adequate: There are answers to questions, but they are not in complete sentences. No or little effort has been made to explore the material further. Some algebra notation may be on the page, but another person can't look at it and understand what it is about.

Good: All questions have been answered in complete sentences. The student shows an understanding of the topic by using both algebra notation and written explanations as appropriate.

Outstanding: In addition to the meeting the criteria for "Good", the student has shown substantial effort to explore the topic on his or her own. The assignment may show an extensive explanation that another student could follow, a brief essay on the topic including additional material found through Internet research, an original picture with written explanations, or an expanded creative effort (such as designing a new number system and showing how it would work).

Recommended Supplies:

1. Human brain (at least semi–functional)
2. Supply of energy for above (example: M & M's, grapes, sour skittles, etc.)
3. Unlined paper
4. Pencil – prefer mechanical pencil 0.7mm with soft grip
5. Large heavy–duty eraser
6. Scissors
7. Ruler
8. Protractor
9. Graph paper (paper with little squares on it instead of lines)
10. Construction paper – just need a few sheets
11. A dictionary, such as http://www.merriam–webster.com. Anytime you are not sure about the meaning of a word, look it up. If you are going to write a college admission test you'll need a good vocabulary, and this course contains some words you may see on the test.

Copying

How to solve a math problem in 10 relatively easy steps

1. DON'T PANIC! Read the problem slowly and carefully. Think of the problem as an interesting puzzle. If you find yourself worrying that you won't be able to solve it, say "So what?" Your ability to solve algebra problems is not a measure of your worth as a person. If you are afraid that being unable to solve the problem means that you are not smart, remember the joke about the smartest man in the world (On board a small airplane are the smartest man in the world, the Pope, and a 6–year old school kid. The pilot comes to the passengers and says, "I have bad news. We're going to crash, and there are only 3 parachutes on board. One of you will have to go down with the plane." Then he grabs a parachute and jumps out. The smartest man in the world says, "Well, I'm the smartest man alive, and I have to pass those genes along." He grabs a parachute and jumps too. The Pope turns to the kid and says, "I'm an old man; I've lived my life. You take the parachute." "Don't worry," the kid says, "we can both jump. The smartest man in the world just took my backpack."). There are some very smart doctors who cannot help their patients because they lack compassion. Some brilliant scientists fail to see the moral implications of their work. Einstein never learned to drive. "Smart" has many dimensions, and is certainly not defined by success in algebra.

2. Decide if the problem has a solution. You are unlikely to encounter an algebra problem that doesn't have a solution, but just being convinced that there is a solution will help you find it. See that it is possible; then do it.

3. Draw a picture or diagram to represent the problem.

4. Deliberately write and draw a little bigger than you usually do. Turn your page sideways if you need a bigger space. Larger numbers look friendlier, and you'll have room to cross out mistakes.

5. Even if you don't know what the answer is, pick a number and plug it into the problem. See what the problem looks like now with a sample value inserted.

6. Find the answer through trial and error if necessary. By the time you do, you'll have a much better grasp of how the problem works. Then go back and use proper algebra methods.

7. Simplify the problem if possible. Instead of solving "Karen walked at a speed of 2.37 miles per hour for 3.2 hours – how far did she walk?", try "Karen walked at a speed of 2 miles per hour for 3 hours – how far did she walk?"

8. Look at the problem from different angles and solve it any way you can. For example, a group of students was trying to measure the dimensions of a box using little wooden cubes. They managed fine with the length and width, but when they tried to measure the height their stack of cubes kept falling over, much to their frustration. An autistic student in the group was not participating but watched quietly as if puzzled by the other students' efforts. Finally he said, "Why don't you just lay the box on its side so the cubes don't fall?"

9. Once you find an answer, consider if it is reasonable. Recall that Sandy took her kids to a fast–food restaurant in the Pretest. She should be a little suspicious if her total bill is say, $189.95 . Once you have what you think is a reasonable answer, check thoroughly to make sure it is correct.

10. See a mistake as a learning opportunity. You learn more from being wrong than from being right. Personally, I learn a lot of things every day .

If you have worked on a problem for a long time without finding the answer, reward yourself with a piece of candy from your supplies. This reinforces the fact that thinking about a problem is just as valuable as finding the answer. Then put your work aside and try again the next day. The quizzes and tests in this course are not meant to be timed.

Chapter 1: The Unknown

What is algebra? To most people, this word means rows of long incomprehensible formulas that have no relevance to their lives whatsoever. Yet when you ask, "What is music?" few people would say that it is pages filled with unintelligible lines and dots that have no meaning. The formulas of algebra are much like sheet music. They are just an efficient way of writing down something that exists in our thoughts. The concepts of algebra, just like the concepts of music, are already in our minds. Complex algebra topics taught in high schools today were discovered independently by many ancient civilizations. Algebra is part of our human heritage, just like music is. Does this sound hard to believe? Let's look at a few examples.

Our first example takes us back thousands of years in time, to meet a hunter named Grom. Grom lives in a primitive village. Schools have not been invented yet, which is lucky for Grom since he is not very bright. Grom and his friend Zorg have just returned from a long and dangerous trip to the ocean, where they have collected shiny shells that they hope will impress certain young ladies in their village. Zorg is talking to a group of people about their adventures, and he has left his pile of shells unattended. As Grom looks at Zorg's pile of shells, he notices that it is larger than his own. Grom can count, of course, but things get a little fuzzy when he gets past three. He does not know exactly how many shells Zorg has, but somehow it seems unfair. As he stares at Zorg's shells, a thought occurs to Grom. He could have more shells if he took Zorg's pile while Zorg wasn't looking.

What is getting Grom into trouble here is his understanding of algebra. He realizes that the number of shells in Zorg's pile is bigger than the number of shells in his own pile. Further, the number of shells in his pile plus the number of shells in Zorg's pile is bigger than the number of shells in his pile now. Notice how many words it takes to express this simple thought. There are better ways of writing the same thing. "is bigger than" can be replaced by the symbol >. Most people find this symbol easy to remember because you put the bigger amount on the side where the symbol is wider. Now it looks like this: The number of shells in Zorg's pile > the number of shells in Grom's pile. Zorg's pile plus Grom's pile > Grom's pile. That is still too long to express anything more than simple understandings. At some point in time, someone realized that if you have an unknown amount you can use a letter to represent it. The letter x has been very popular for this purpose, but really any letter will do. By convention we usually use lower case letters. So, to save ourselves some writing we can put z instead of "the number of shells in Zorg's pile", and g instead of "the number of shells in Grom's pile." Now we can write Grom's thoughts down quickly: **$z > g$** and **$z + g > g$** [Oooo, scary–looking algebra!].

[Note: You are about to cross a stop line. That means you should think about what you just read. If the idea is not clear, read the section again.]

Let's leave Grom to solve his moral dilemma by himself, and travel forward in time to a medieval castle somewhere in Europe. Lord Seston has just discovered that his wine cellar is now empty because his supplier hasn't been paid. Furious, he sends for Thomas, his accounting scribe. Thomas explains that the money shortage is due to the difficulty in predicting income from taxes when it is not known how many peasants are currently on the lord's lands. Thomas continues on about the need to count the peasants often, and how many peasants there would have to be to support various expenses. Lord Seston was never very good with numbers, and he soon finds himself getting a nasty headache. "I don't care how many of those stupid peasants there are! Just tax each one double and I'll have twice as much money," he yells. Notice that Lord Seston is not about to be stopped by an unknown quantity such as the number of peasants [or by compassion either for that matter].

Algebra allows us to continue reasoning when an unknown number is involved. What Lord Seston realizes is that his income from taxes is equal to the number of peasants times the average tax that a peasant has to pay. Let's use the letter i instead of "the income from taxes", so we can write: i = "number of peasants" x "average amount of tax a peasant pays". We can make this even more compact if we replace "number of peasants" with the letter n, and "average amount of tax a peasant pays" with the letter t, like this: **$i = n \times t$**. We know from arithmetic that there is something predictable about multiplying two numbers. If you make one of the numbers twice as big, the answer will be twice as big. For example, 5 x 4 is twice as big as 5 x 2. When the average tax paid is twice as high, the income from taxes doubles. Unfortunately for Lord Seston however, income still also depends on n, the number of peasants. Six months later the number of peasants has mysteriously decreased, while the population of the surrounding areas has gone up...

Next, let's look at another unknown and rather uncooperative quantity. Becky is getting some flea medicine for her cat, and she has to know its weight. She puts the cat on her bathroom scale, but no matter how hard she tries her cat won't sit still. To get around the problem she picks her cat up and stands on the scale with it. Now the scale reads 138 pounds. Next she puts the cat down and weighs herself. Becky's weight is 127 pounds. Becky concludes her cat weighs 11 pounds. This is a simple algebra problem that few people seem to have trouble with, until you put it in scary algebra notation. Before we start weighing, we have two unknown quantities: Becky's weight, and the cat's weight. The most appropriate letters to use would be b for Becky's weight, and c for the cat's weight. The facts are still the same: Becky and her cat together weigh 138 lbs [**b + c = 138**]. Becky weighs 127 lbs., which we can write as **b = 127**. After looking at this for a bit, we write: 127 lbs plus the cat = 138 lbs or **127 + c = 138**. Looking at this a little longer, we see that c has to be 138 – 127, which makes it 11. **Notice that the units that go with the problem, pounds in this case, are not included in our algebra notation.** This keeps our notation simpler when there is only a single unit involved. After we conclude that **c = 11**, we have to go back to the problem and say that the answer is 11 lbs.

Algebra can tell us more about an unknown quantity, and help us find the exact value. Sometimes there are two or more correct values for an unknown. Other times we may end up with a relationship between two different quantities, which can turn out to be very useful. Since the universe is full of unknowns, people, and especially scientists, have a frequent need for algebra. When the problems are simple you can just think them through, but when they get more complicated it actually helps to use letters to represent the unknown, and write complex–looking equations and formulas. Colleges want you to be able to do this. They need to know that you are an educated person, and that you will have the basic skills required to take college math and science courses. Employers are unlikely to ask about your knowledge of algebra, but when basic algebra problems come up in the workplace you will be expected to handle them and get the right answer.

Algebra has a reputation for being difficult. Is this true? Well, unless you are smarter than the rest of us, you're going to have to strain a little to be able to understand the more complex parts. Fortunately, a little mental straining doesn't hurt us one bit, just like it doesn't hurt us to get some exercise. Your brain improves with use, just like your muscles do. What does hurt is to look at your algebra course and say "I don't understand this; I must be stupid." Understanding is the whole point of algebra, and it often doesn't come instantly. Throughout history, mathematicians have worked at these problems, often spending days or months [or sometimes even years!] to understand a single concept. And no, these were not stupid people. Eventually all of their efforts were put together in an algebra book. The book tries to help readers by explaining how things work, but not everyone thinks the same way. In fact, smarter people often think differently or want a deeper understanding. Sometimes you end up having to start right from the beginning to find your own explanation, and that just takes some effort. If you don't get it today, sleep on it and try it again the next day, and the next if you have to. Tell yourself: "I'm getting smarter just by thinking about this" .

Review Exponents and Order of Operations

An understanding of exponents is necessary in order to learn algebra. If you are not comfortable with this subject yet you can read about it in the Arithmetic Review e-book, or do an internet search for “exponents” to get the basics.

Performing operations in the right order is essential in arithmetic.   You may have seen this phrase before: "**P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally", where **P=** parentheses, **E =** exponents, **M =** multiplication **D =** division, **A =** addition, and **S =** subtraction.

In algebra an understanding of this order is critical because things can look a lot more confusing when an unknown quantity is involved.

Have a look at an explanation written for this section by Lexi, and you'll see that there is already a bit of a glitch when dealing with parentheses:

"I know that most of you know all this by now but this is for people that are having trouble with it. So enjoy.....!

I practiced exponents and the order of operation. Exponents are simple to learn. For example, 12^2. This shortened number is really 12 x 12 which is 144. A bigger number with an exponent looks complex but it really isn't. Like, 123^2 is just 123 x 123 =15129. Actually I had to figure it out on a piece of paper, but if it is only a piece of paper you need then go for it. It is not a hard lesson for me to learn at all.

The order of operations is harder than learning exponents but if you understand * **P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally * you should have no problem. Here is an example:
($5^2 + 2^4$) + 5 x 2 = ?

First, we need to know **P**lease which is **P**arenthesis. But we can't do the parenthesis unless you do the **E**xponents which is **E**xcuse. Our equation would end up being: (25 + 16) + 5 x 2 = ?. Now, we need to do the **P**arenthesis which would end up being: 41 + 5 x 2 = ?. The next step would be **M**y which is **M**ultiplying. So it would end up looking like this: 41 + 10 = ?. Now I am sure you could figure it out on your own now but you should still know **D**ear **A**unt **S**ally. **D**ear is **D**ivision but there is no division in this equation so you can skip that. Then there is **A**unt which is **A**ddition. We need to add to get our answer so: 41 + 10 = 51. We have already solved our equation but we still need to know that **S**ally stands for, **S**ubtraction. Now we know that:
($5^2 + 2^4$) + 5 x 2 = 51.
I hope that helped."

Thank you Lexi; that does help. Do remember that multiplication and division actually have the same priority, so they are done from left to right:

10 ÷ 2 x 5 = 25

The same goes for addition and subtraction. Simply do them in the order you see them.

Working with Parentheses

From the previous example you can see that you can't always just "do parentheses first". Now we are going to start thinking about parentheses a little differently. Imagine that the parentheses are saying: "The whole thing inside of here." So, 5 + (3 + 4) means 5 + "the whole thing inside the parentheses", which is 7. We do 5 + 7 = 12. Here it is perfectly fine to just remove the parentheses, since 5 + (3 + 4) is the same as 5 + 3 + 4.

10 – (3 + 2) means 10 minus "the whole thing inside the parentheses", so we say 10 – 5 = 5. That is **not** the same as 10 – 3 + 2. If you needed to get rid of the parentheses you would write:

10 – (3 + 2) = 10 – 3 – 2

Try to think of this as the parentheses telling you, "10 minus 3, and also minus 2".

And what about 10 – (3 – 2)? We can see that it is 10 – 1 which is 9, but if we just tried to remove the parentheses we would end up with 10 – 3 – 2 which would be 5. Well, while there are no unknowns around we can just leave those parentheses until we finish adding or subtracting what is inside of them. In the next chapters we'll learn more about how to safely remove parentheses if there are unknown quantities involved.

Practice using order of operations rules at http://www.math.com/school/subject2/lessons/S2U1L2GL.html#sm1. Be careful not to go past the "Order of Operations" section.

If you have never seen the material in this course topic before, you should have spent at least a week exploring it. The point is to learn something rather than finish quickly. If you start the quiz and then realize that you don't understand things, just leave the quiz and go back to study the material some more. Each quiz does have a few questions that don't have obvious answers, to help you think about the subject a little more. However, a score below 70% may indicate that you have not spent enough time preparing. If you do not know how to calculate your quiz score, you need to look at the Arithmetic Review e-book.

Chapter 1 Quiz

There are signs other than < and > that are useful to compare amounts. We use = to indicate that two quantities are the same, and ≠ to show that they are not equal. Other useful signs are ≤, which means is less than or equal to, and ≥, which means is bigger than or equal to. The following problem uses one of these signs.

1. Larry, Sam, Byron and Jake step into a service elevator. The elevator is designed to carry a maximum load of 1000lbs. An alarm sounds, indicating that the maximum load has been exceeded. Sam offers to get off, but Larry tells him to wait. Larry has spotted a large cockroach on the elevator floor and chases it out into the hallway. Oddly enough, now the doors close and the elevator goes up. Using the letters of everyone's first name to represent their weights, and c for the cockroach's weight, which one of the following statements is true:

a. $\ell + s + b + j + c > 1000$

b. $\ell + s + b + j + c \leq 1000$

c. $s > 250$

d. $\ell + s + b + j > 1000$

2. Sandra's wallet is empty. She goes to an ATM and takes $200 out of her account. Three days later, she only has $10.50 left in her wallet. Using m to represent how much money Sandra spent these last three days, what would be the **best** way to express this situation in algebra notation?

a. 200 + m = 10.5

b. 200 – m = 10.5

c. $200 – $m = $10.50

d. "How could I have spent $m that fast?"

e. $10.50 – $m = $200

3. The main purpose of using algebra notation is that it:

a. gives you the answer

b. makes complex problems easier to work with

c. makes you look smarter

d. makes all problems easier

4. 8 + 40 ÷ 8 – 2 x 5 – 3 =

5. 20 – (6 + 4) =

6. 35 – (7 – 3) =

7. The actual total number of shells that Grom and Zorg collected was 34. Write this fact in algebra notation, using g for the number of shells Grom collected, and z for the number of shells Zorg has.

8. True or False: 3 + 4 x 5 < 24

9. We have three numbers, but we don't know what they are. We will call the first number a, the second number b, and the third number c. The following information is available about these three numbers: $a < b$ and $b < c$. Based on this information, we can conclude that:

a. $c < b$

b. $a > c$

c. No conclusion can be drawn because the numbers are unknown

d. $a < c$

e. $a = c$

10. True or False: $4 \geq 4$

11. $10^4 =$

12. $19^1 =$

13. $5^2 + 5^3 =$

14. Mike and Ike are sharing a box of candy. Because there are an odd number of candies in the box, Mike ends up with one more candy than Ike. The following statements use m for the number of candies that Mike has, and i for the number of Ike's candies. Mark each statement as true or false:

a. $m \neq i$ T or F

b. $m < i$ T or F

c. $i < m$ T or F

d. $i + 1 = m$ T or F

e. $m + 1 = i$ T or F

Help for Quiz

Help for selected questions is provided here. Answers are located in the answer key at the end of this e-book.

Question 1: This problem may seem a little confusing at first glance. What we know is that the elevator will carry a load that weighs 1000 pounds or less. An alarm will sound if the load is too heavy. In this case the alarm sounds, so the total weight in the elevator must be more than 1000 pounds. At first it seems like there are just 4 people in the elevator, but then we find out that there are 4 people and a bug. Once the cockroach is gone the elevator goes up, so the load is now 1000 pounds or less. The 4 people together don't weigh more than 1000 pounds. The 4 people and the cockroach do weigh more than 1000 pounds. Read the choices carefully and select a statement that is supported by the facts in the problem.

Question 4: Remember to do these operations in the correct order. Multiplication and division get priority over addition and subtraction.

Question 10: The sign ≥ means bigger than *or* equal, not bigger than and equal. Only one of these conditions has to be satisfied to make the statement true.

Number Systems

You probably feel pretty confident that you know your basic math, such as simple addition, subtraction, times tables etc. Yet your whole life you've been taught just one number system. Our number system is based on the number 10, and we handle everything through groups of 10. If you were moving to a planet where everyone had 6 fingers on each hand instead of 5, you'd find yourself having to learn math all over again by counting in groups of 12. Humans have used different number systems in the past. The ancient Babylonians actually had a base 60 number system!

People thought about computers even before the parts required to make them were invented. The first efforts involved complex mechanical machines. Once we had electronic parts people wasted no time trying to build a computer with them, even though there were no transistors yet and they had to use bulky vacuum tubes. At first it seemed simple enough to use electricity to simulate numbers. Just have no current for the number 0, a little more for the number 1, a bit more for the number 2, and so on up to 9. Unfortunately this didn't work out. The current could not be adjusted accurately enough, and sometimes an 8 might end up as a 9 for example which made the whole system useless. The engineers eventually realized that they could represent only 2 numbers, by having the current either on or off. This must have seemed like a big problem at first. After all, if you just have two numbers, how much can you do with them? We could have 1, 2, 11, 12, 21, or 212 etc., but what use is that? Fortunately people had already thought of something else: you can use 0 and 1 instead of 1 and 2. That may not seem better, but by using 0 and 1 we can make "nicer" numbers like 10, 100 or 1000. A lot of digits are still missing, but we can now start counting. We have a number for 0, and a number for 1. After that there are no digits left, so we have to go to 10, which now represents 2. From there you go to 11, which represents 3. Again you are out of digits, so now the next number is 100, which represents 4. Carefully count a bit further: 5 must be 101. This base 2 system is called binary, and it was actually invented long before computers arrived on the scene. The root word bi means 2, like bicycle – a contraption with two wheels.

This should make you think about how our regular number system is set up. We use 10 separate digits, from 0 to 9. Past 9, you have to re–use digits, so we use a new place – the tens place. This allows us to count past the number 9. Once we get past 19 we put a 2 in the tens place, and again start counting at 0: 20, 21, 22, ... We can go all the way to 99 before we again run out of digits. Now we need the 100's place. Put a 1 there and you can start over again.
Of course when you have only two digits to use, this process goes a lot faster. We can now only count to 1 before we are out of digits and need another place. We don't call that place the 10's place, because it represents 2 rather than 10. We just put a 1 in this 2's place and start over: 10, 11, and now we are out of digits again. The next place we use is the 4's place, because 100 represents the number 4. Counting further, we get 100, 101, 110, 111 and that's it before we have to add another place – the 8's place.

Follow the links on the next page and see if you can play around with this stuff. It is really a whole new way of looking at numbers, and it takes some getting used to. Be patient – this course was made to allow plenty of time to explore new concepts like this on your own. Learning is more than just sitting in a classroom waiting for someone to tell you everything. If you realize that you can learn on your own you'll be way ahead of the game.

So, where do we go from here with this new number system? Well, we have 100 which is 4, and 10 which is 2, so let’s add those up:

110 = 6

To get to 7, all we have to do is add 1:

111 = 7

Because we have no other digits, we now have to use the next available place value:

1000 = 8

Hmm, I'm noticing something here. In our regular number system, 10, 100, 1000 etc. are all powers of 10 (10, or 10 times 10, or 10 times 10 times 10, and so on). In this new number system 10 is 2, 100 is 4 (2 times 2), and 1000 is 8 (2 times 2 times 2). I would guess that 10000 will turn out to be 2 times 2 times 2 times 2, which is 16. See if you can count up carefully until you get to 16.

Binary is fun to explore, and you'll see that it is remarkably efficient. Check out this link to see how it works: http://www.swansontec.com/sbinary.htm.

At first glance it may seem like binary numbers are hopelessly long, but with binary you can use your fingers to show numbers up to 1023 [Put your finger down for 0, and up for 1]. This video shows you how to count in binary on your fingers: https://www.youtube.com/watch?v=apCLHmPsC68

Once you understand binary, you may like it so much that you want to get yourself a binary clock. Look for one online to see how it works.

Be sure to take your time exploring this topic. If you need to work with it again five years from now, will you remember it? If it interests you, also read about the hexadecimal counting system. This amazing number system uses 16 different digits! Because we really have only 10 digits, it counts like this: 0, 1, 2, 3, 4, 5 6, 7, 8, 9, A, B, C, D, E, F, 10. That makes it a base 16 system, and it is used in computer coding to provide a more manageable translation of the underlying binary. Next time your computer complains of an error in module 0x8000FFFF, you'll be able to say, "Hey, I know that, that's hex." The situation will be just as bad, but you'll feel better only being annoyed rather than both annoyed and confused.

Hexadecimal is also used to code for colors.

How Hex Code Colors Work – and How to Choose Colors Without A Color Picker (freecodecamp.org)

Look at a hex color picker online. What is your favorite hex color?

Binary Quiz

1. Byron is 17 years old. What is his age in binary?

2. Here is a binary number: 101. What is the decimal equivalent number?

3. You can add binary numbers just like you add decimal numbers with multiple digits. What is 1011 + 111? Write the second number underneath the first, and use carrying. The answer you get will be in binary. Then translate the two numbers and the answer into decimal to check your work.

4. What is 1100 – 101? Use borrowing carefully, and check your work.

Portfolio Chapter 1

Here are some ideas to get you started:

For the first chapter you could take some example from your own life and express it in algebra notation, calculate your age in binary, and translate the hex code of your favorite color into decimal. You might also want to design your very own number system using a base that you think is best.

Chapter 2: Negative Numbers

An Anti–World?

Everything in our world is made of matter, but there is also something called antimatter. The atoms that make up matter consist of negatively charged electrons surrounding a nucleus with positively charged protons and neutrons that don't have a charge. Using the techniques of algebra, a physicist named Paul Dirac predicted the existence of a positively charged anti–electron, the positron. Today we know that positrons really exist and we can actually create anti–atoms of hydrogen that have a positron surrounding an anti–proton. It is extremely difficult to make antimatter because matter and antimatter annihilate each other when they touch. So what kind of container will you use to hold your antimatter? Actually, scientists use magnetic fields to try to confine these anti–atoms, and they can only make a few of them. If we were able to create a significant amount of antimatter we could combine it with matter to generate large amounts of energy.

If there is antimatter, is there also an anti–universe? So far we have not been able to detect much antimatter out in space, but scientists are looking for it. The Big Bang should have created antimatter as well as matter, so where did it go? If it is in an alternate universe, what would such a universe be like? Maybe it would be parallel to ours, with an anti–copy of everything that is in our universe, even ourselves. Or maybe its time would somehow run in a direction opposite from ours so that we could never find it. That might be for the best anyway, since we could never enter such a universe without annihilating ourselves!

Negative Number Math

Sometime in the distant past, man invented money. Shortly afterwards, man invented debt. This was the beginning of our discovery of negative numbers. Negative numbers are truly amazing, and they are a natural part of our universe. There are positive and negative electrical charges, and there is matter and anti–matter. We can work with negative numbers just like we do with the positive ones. It usually helps to think of positive numbers as money you have, and negative numbers as debt, or money you owe. This way, you should have less trouble adding or subtracting them.

Let's start with something simple:

$10 - 12 = ?$

If you have \$10, and then spend \$12, you end up \$2 in debt:

$10 - 12 = -2$

Most people don't have a problem understanding that. If you spend more than what you have, you end up with a negative amount of money. What is a little more difficult is when you start out with a negative amount, and then subtract from it:

$-3 - 4 = ?$

You start out by being \$3 in debt. Next you spend another \$4. \$4 is subtracted from the total amount of money you have. Now you are \$7 in debt, which means that you have −\$7.

$-3 - 4 = -7$

Just remember that whenever you subtract, things get more negative. All that spending just adds to your debt.

What is $-3 + 4$?

If you are \$3 in debt, and you get \$4, you can use this money to cancel out your debt, and have some left over. $-3 + 4 = 1$.

Sometimes you will be asked this same question the other way around: what is $4 + -3$? Now you have \$4, and you get a \$3 debt. When you add these things together you again end up with \$1. $4 + -3 = 1$. Notice that $4 + -3$ is the same as $4 - 3$.

It is also possible to subtract a negative number. Subtracting debt is removing debt, which is really the same as gaining money. You end up becoming wealthier. If you subtract \$4 of debt, you are really adding \$4:

$-3 - -4 = -3 + 4 = 1$

Subtracting a negative number results in addition: − − = +

Now we can fix that little problem we saw earlier with the parentheses: 10 − (3 − 2) = 9. Here we have to subtract the whole thing inside the parentheses. Imagine the parentheses saying, "10 minus 3, and also minus −2". We subtract 3, and also subtract −2:

10 − 3 − −2, which is the same as 10 − 3 + 2 = 9.

$10 - (3 - 2) = 10 - 3 + 2$

Before exploring multiplication and division with negative numbers, we should consider how these operations work when we use them with regular positive numbers. Remember that a fraction is really a division, and each division has a multiplication that goes with it:

$\frac{6}{3} = 2$ (6 divided by 3 equals 2) and $2 \times 3 = 6$

$\frac{6}{2} = 3$ (6 divided by 2 equals 3) and $3 \times 2 = 6$

Armed with these simple facts, and some common sense, we can work out the rules for multiplying and dividing with negative numbers.

Three times a \$20 debt is a \$60 debt:

$3 \times -20 = -60$

That makes sense, and it gives rise to the rule that **a negative number multiplied by a positive number is a negative number**. Now what about 20 x −3? Well, if you owe 20 people \$3 each you're still in just as much trouble [or even more since they may all complain to you].

$20 \times -3 = -60$.

So, a **positive number multiplied by a negative number is also a negative number**.

In the same way that you divide some candy among your friends, you can divide debt [it just won't do quite the same thing for your popularity]. For example $60 in debt divided among 3 people works out to a $20 debt for each person.

We can write that as: $-60 \div 3 = -20$.

The rule that goes with this is that **a negative number divided by a positive number gives a negative number.**

Division can be rearranged: if $6 \div 3 = 2$, then $6 \div 2 = 3$. $\left[\frac{6}{3} = 2 \text{ and } \frac{6}{2} = 3\right]$

So, if $-60 \div 3$ is -20, then

$-60 \div -20 = 3$

$\left[\frac{-60}{3} = -20 \text{ and } \frac{-60}{-20} = 3\right]$

A negative number divided by a negative number equals a positive number.

Next, we will tackle the idea of dividing a positive amount, like $60, by a negative number:

$60 \div -3 = ?$

Now where are we going to find negative 3 people?? No need to worry; if you are offering to divide up $60 you can find people willing to take it anywhere 😁. So, whenever I divide $60 among 3 people in our universe, somewhere in a parallel, anti–universe, 3 lucky people will also each receive $20. These people are unaware that their whole world is made up of anti–matter. Some of them may know that their scientists are attempting to create significant amounts of matter, just like our scientists attempt to create anti–matter. But that is a subject for physics class. Anyway, when I go to record the amount of money each anti–person receives, I realize that this is not money in our universe. From my point of view this money exists only in an anti–universe, so it is actually anti–money. I record the amount as –$20. Therefore, 60 ÷ −3 = −20.

You will not find this simple explanation in any math textbook. All you'll see is a rule that says that **a positive number divided by a negative number will be a negative number**. [This is obviously the result of a government conspiracy to cover up the existence of a parallel anti–universe.]

Kidding aside though, if you stop and think about it, there are only two candidates for the right answer to 60 ÷ -3. Either $\frac{60}{-3}$ = 20 or $\frac{60}{-3}$ = −20. Negative numbers represent the same size amounts as positive numbers, only in the opposite direction. Division and multiplication follow the same arithmetic rules, and only the signs change. If $\frac{60}{-3}$ was equal to 20 then 20 x −3 would have to be 60. But we already saw that if you owe 20 people $3 you end up with a $60 debt instead of positive $60. $\frac{60}{-3}$ has to be −20.

What happens when you multiply two negative numbers? Because $\frac{60}{-3}$ = −20, that means that −20 x −3 has to multiply back out to 60.

−20 x −3 = 60

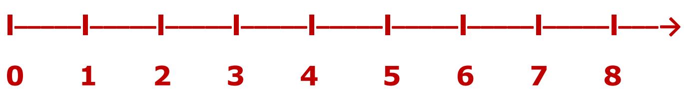
A negative number times a negative number equals a positive number.

Be very careful when applying order of operations to a negative number with an exponent. -7^2 is −(7 x 7) which is −49, NOT −7 x −7 which would be 49. The − sign is really a multiplication by −1 since −7 = 7 x −1. In the order of operations the exponent comes first, so raising to the second power is done before applying the minus sign. **If you have a pressing need to indicate −7 x −7 you would write it as $(-7)^2$**.

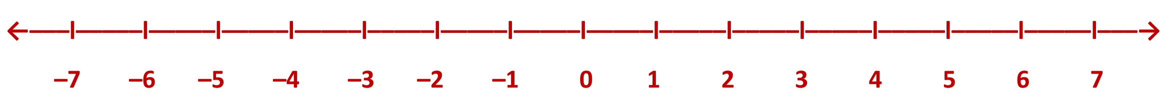
The square root of a number is some number that when multiplied by itself, gives that number. 5 times 5 is 25, so 5 is the square root of 25. −5 times −5 is also 25, so there is a negative square root of 25. To avoid confusion, when we write $\sqrt{25}$ it is agreed that we mean the positive root which is 5. To refer to the negative square root, we write $-\sqrt{25}$.

If you go back over the rules for multiplication by negative numbers, you'll see that something is missing. 1 x 1 = 1, and −1 x −1 = 1. There is no way to get $\sqrt{-1}$. That has never stopped people from trying, and eventually [around the 16th century] they came up with an entirely new set of numbers, the imaginary numbers, so that i x i = −1. 2i x 2i = −4. We might say that people "invented" these "imaginary" numbers. Yet today electrical engineers and other people working on practical problems find that they can't manage without them. Wouldn't it be more accurate to say that we discovered the imaginary numbers rather than invented them? Algebra will take you into a whole new dimension as you explore these numbers in Algebra II.

The Number Line



This is the number line as it existed before people officially recognized negative numbers. The arrowhead on the right side indicates that the line goes on forever. The interesting thing about the number line is that you can pick any point on it, let's say 4, and then move to the right to do addition. If you want to do 4 + 4, you start at 4 and move 4 units to the right to end up at 8. For subtraction you move to the left, so if you're doing 4 – 3 you start at 4 and count 3 units to the left, to end up at 1. That would be helpful if you really had a hard time doing addition and subtraction, which most people don't until they get to the negative numbers.



This is the number line as we see it now. To do a problem like 4 – 7, start at 4 and move 7 spaces to the left (the negative direction). You end up at −3. If you find it hard to subtract a negative number from a negative number, like −2 – 4, try using a number line. Because you always move to the left to do subtraction, you can see that you end up at −6 if you subtract 4 from −2.

By definition, the numbers on the number line get smaller as we move from right to left. This means that −7 is a smaller number than −2, or −7 < −2. Sometimes there is something about this rule that doesn't feel quite right. For example, if I have −$7, that is a bigger debt than if I have −$2. −7 may be a smaller number, but is a "bigger" negative quantity. When you look at the number line, you can see that −7 is just as far away from 0 as 7. To account for this, we have something called **absolute value**. If you just want to compare the relative size of numbers, regardless of whether they are positive or negative, look at their absolute value. The absolute value of −7 is 7, and the absolute value of 7 is also 7. **Absolute value is always a positive quantity, or zero.** It is written by putting the number between two lines, like this:

$|-7| = 7$

Now we can indicate that −7 is a "bigger" negative number than −2: $|-7| > |-2|$.

Movie from Khan Academy

The following movie explains negative numbers using a number line.

http://www.youtube.com/watch?v=C38B33ZywWs

If you liked this this movie, you may enjoy others from Khan Academy later on in the course. These movies provide simple "how to" instructions. Make sure you also understand why things are done this way by reading the chapters in this course.

Summary

Addition or Subtraction

$10 - 7 = 3$

$7 - 10 = -3$

$-5 - 4 = -9$

$8 + -4 = 4$

$8 - - 4 = 8 + 4 = 12$

Multiplication

$2 \times 3 = 6$

$2 \times -3 = -6$

$-2 \times 3 = -6$

$-2 \times -3 = 6$

Division

$6 \div 3 = 2$

$-6 \div 3 = -2$

$6 \div -3 = -2$

$-6 \div -3 = 2$

Absolute value is never negative: $|4| = 4$ and $|-4| = 4$

Practice

Go here to practice adding and subtracting positive and negative numbers: http://www.funbrain.com/cgi-bin/nl.cgi?A1=s&A2=4

Search online to find practice problems for multiplication and division with negative numbers. Once the problems seem easy, start creating your portfolio page.

Portfolio Chapter 2

In your own words, write what you learned about negative numbers. What are all the rules? Do they make sense to you?

What is your feeling about negative numbers? Are they really part of our universe, or just something we made up for convenience? The mathematician Leopold Kronecker [more about him later] said that God created the positive integers (whole numbers), and everything else is man–made. Think of reasons to support or contradict his statement. Note that since this is not public school, you are welcome to consider this question from a religious point of view if you want. Otherwise you may like the argument that positive whole numbers are more "natural" and less abstract.

As you progress through your teens, your brain starts to engage in more abstract thoughts and you may find negative numbers more appealing. So if you don't like them now, try giving it some time and maybe they'll grow on you, sort of like spinach...

Chapter 2 Quiz

Draw a number line that includes positive numbers to 8 and negative numbers to −8. Put an arrowhead on the left tip of the line to indicate that the numbers get smaller forever, and also an arrowhead on the right side. Use your number line to solve the first 5 questions of the quiz.

1. $4 - 7 =$

2. $6 - 12 =$

3. $-3 + 5 =$

4. $-2 - 6 =$

5. $-7 + 14 =$

6. True or False: $-9 > 0$

7. True or False: $|-100| < 100$

8. True or False: $|-2| > -3$

9. True or False: $-4 < -3$

10. True or False: $|-2| \geq 2$

11. True or False: Adding negative numbers to the number line expands our horizons. The number line is now twice as long as it was before.

12. $-3 \times 4 =$

13. $5 \times -3 =$

14. $-25 \times -4 =$

15. $9 \times -10 =$

16. $-1 \times -1 =$

17. $-30 \div -6 =$

18. $100 \div -5 =$

19. $71 \div 71 =$

20. $\frac{-50}{25} =$

21. Express in proper algebra notation: p is a positive number, or zero

a. $p \geq 0$

b. $p > 1$ or $p = 0$

c. $p = +$ or 0

d. $p > 0$

22. Express in proper algebra notation: n is a negative number

a. $n = -$

b. $n \leq -1$

c. $n < -1$

d. $n < 0$

Check your answers in the answer key. Every wrong answer is an opportunity to learn something, and you may want to add that to your portfolio page.

Chapter 3: Making Algebra

When I was young our family didn't own an oven, so I thought cakes came from bakeries. After I moved away from home I bought packages of prepared cake mix to make in my own oven. I was quite surprised to find out later that you can actually make a cake from scratch, with flour and sugar and stuff, and it tastes better too! You can even experiment with the recipe to change the flavor. Just like I used to get cake from a box, I used to get my algebra from math books. Then I discovered that you can actually make your own algebra. It's still math, but it isn't nearly as boring. Let's start with an easy example. The ones times table is very simple. 1 times 1 is 1, 2 times 1 is 2, 3 times 1 is 3, and well, you know how it works. But because you know how it works you can use an unknown number: a number times 1 is that same number. Now make that look fancy by writing "n" instead of "a number":

$$n \times 1 = n$$

That makes sense. Even though we don't know the value of the number n, we can still say that n times 1 is equal to n. The point of algebra is to deal with unknown quantities, so we have to find ways to do that. Let's see what else we can make. I know that when I multiply a number by 0, the result is always 0:

$$n \times 0 = 0$$

Before you read on you may want to create some algebra yourself. It really isn't that hard because there are many things that always work out the same way.

When I add 0 to any number the number doesn't change:

$$n + 0 = n$$

And if I divide a number by 1 it doesn't change either:

$$n \div 1 = n$$

Dividing a number by itself also has a predictable result: $6 \div 6 = 1$ and $100 \div 100 = 1$.

$n \div n = 1$

Subtracting a number from itself always results in 0:

$n - n = 0$

Adding or multiplying two numbers gives the same result when you switch the numbers around. 3 + 5 = 5 + 3, and 4 x 6 = 6 x 4. Now there are two numbers involved, so you will need two unknowns. I will call the second number m:

$n + m = m + n$

$n \times m = m \times n$

That doesn't work with subtraction, but consider what happens when you switch the numbers around:

$n - m$ or $m - n$

Try it with some actual numbers. Do you see a pattern?

Absolute value signs can take care of that:

$|n - m| = |m - n|$

There is something about addition and multiplication that allows us to create times tables. 3 x 4 is really 3 groups of 4. To find the answer to 3 x 4 = __ , we add 4 + 4 + 4 = 12. In fact, 3 x any number is the number + the number + the number:

$3 \times n = n + n + n$

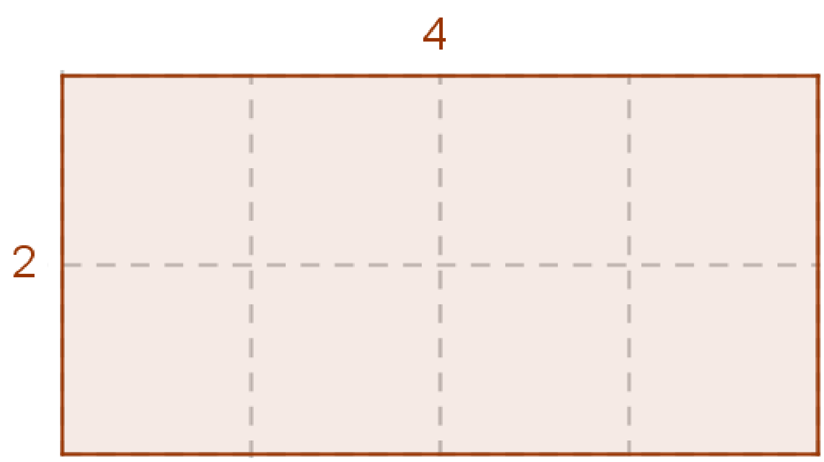
4 + 4 + 4 + 4 + 4 is 5 groups of 4, or 5 x 4, so

$n + n + n + n + n = 5 \times n$

Advanced algebra can look complex and scary, but it was made from these easy rules. If you understand arithmetic you can understand algebra.

Creating Formulas

One important use of algebra is to create recipes for doing things. The rectangle below has one side that is 2 units long, and one side that is 4 units long. This means that we can fit 2 rows of 4 square units inside the rectangle. We say that the area is 2 x 4, or 8 square units



We can show our understanding of how to find the area by making a general recipe: "To find the area of a rectangle, multiply the length times the width." We write these recipes out as formulas:

$A = \ell \times w$

This formula contains 3 unknowns. The A stands for area, ℓ is the length, and w is the width.

People work from specific examples to eventually make general formulas. That started thousands of years ago, with practical problems like these: If I can trade 4 goats for one cow, I need 12 goats to get 3 cows. Even back then, people were able to see the basic idea behind this problem. You multiply the number of cows you want by 4 to come up with the number of goats you need. The ancient civilizations of the Babylonians and the Egyptians developed very impressive general ways to manage complex problems. However, the first use of a letter to represent an unknown quantity apparently did not occur until around the year 200, and it really didn't catch on in a big way until the Renaissance. Combined with the re–discovery of ancient mathematics, the regular use of letters to represent unknowns allowed for fairly rapid developments. The Renaissance started in the 14^{th} century, and by the middle of the 17^{th} century things had progressed far enough to set the stage for the discovery of calculus. When you take required middle and high school math courses you will be retracing the journey of ideas and discoveries from ancient times to this particular point.

Patterns

One of the few occasions when you may actually be asked to come up with your own formulas is to show that you understand patterns. Seeing the pattern is usually easier than actually making the formula, but there are a few tricks you can use.

1. Linear Patterns

These patterns are called linear because the points lie in a straight line if they are placed on a graph.

2, 4, 6, 8, 10, …

The … sign means that that things just continue along in the same pattern.

Most students will guess that the next number in this pattern would be 12. That is correct, because this sequence of numbers is built on the pattern of adding 2 to the previous number. Because it starts with the number 2, each number in the pattern is a multiple of 2. We can make the pattern like this:

1 x 2 = 2

2 x 2 = 4

3 x 2 = 6

4 x 2 = 8

5 x 2 = 10

The 6^{th} number is made by 6 x 2. The 100^{th} number would be 100 x 2. Using the ideas of algebra, we say that the n^{th} number is n x 2 (where n is a whole number). The formula that we can make for the pattern is: The n^{th} number is 2 x n. We can check that the formula is correct by substituting numbers for n. When n = 1 the formula reads 2 x 1, which is 2. When n = 2 the formula gives 2 x 2 = 4, and so on.

Now try another pattern:

3, 6, 9, 12, 15, …

A formula for this pattern would be: The n^{th} number is 3 x n.

Here is a more interesting pattern:

6, 11, 16, 21, 26 …

If you look carefully at these numbers, you will see that they go up by 5 each time. Whenever you are looking for a pattern, it is a good idea to write the difference between each two numbers below the sequence:

6 11 16 21 26
5 5 5 5

Unfortunately we cannot make this pattern by using the formula 5 x n, because that gives us 5, 10, 15, 20. However, with a little ingenuity we can tweak our formula. The pattern we want starts at 6 instead of 5, so 5 x n + 1 will give you what you need.

In the same way, the sequence 2, 6, 10, 14, 18, … can be made from the formula 4 x n – 2.

To see what to add or subtract, just look at the first term, where n = 1. For the pattern created by 4 x n, the first term is 4. If we actually want 2 to be the first term, all we have to do is subtract 2: 4 x n – 2. To check your formula, you could imagine what the number before the first term would be. That would be the 0^{th} number, so n would be 0. For 2, 6, 10, 14, … the 0^{th} number would be –2. The formula 4 x n – 2 does in fact give you -2 when you pick 0 for n.

Some patterns are made up of decreasing numbers:

9, 6, 3, 0, -3, …

The numbers decrease by 3 each time. Just use -3 in your basic pattern: $-3 \times n$. When n is 1, that starts at -3, but we really want the first number to be 9. Simply add 12 to your formula: $-3 \times n + 12$.

2. Square Patterns

While many patterns have a constant difference between consecutive numbers, others show an ever increasing difference, like the sequence 1, 4, 9, 16, 25, …:

1 4 9 16 25
3 5 7 9

Looking at the difference of the differences, we see that those differences are increasing regularly:

1 4 9 16 25
3 5 7 9
2 2 2

This is characteristic of a pattern made from square numbers.

You can use the formula n^2 to make the pattern. If you are only asked to find the next number rather than a formula, you can follow the pattern of the differences. The sequence starts with 1. The difference between 1 and 4 is 3. The difference between 4 and 9 is 5, the next difference is 7, and then 9. Because the differences are going up by 2 each time, the next difference should be 11, which makes the next number 36:

1 4 9 16 25 36
3 5 7 9 11
2 2 2 2

2, 8, 18, 32, … is a sequence with the same pattern of differences. Its formula is $2 \times n^2$.

2 8 18 32
6 10 14
4 4

Again, you can just find the next number by seeing that the difference between 2 and 8 is 6, between 8 and 18 it is 10 and between 18 and 32 it is 14. Because those differences are increasing regularly the next difference should be 18, which makes the next number in the sequence 50:

2 8 18 32 50
6 10 14 18
4 4 4

3. Cubic Patterns

If you have to look at the difference of the differences of the differences before you see a regular change, chances are you are dealing with a pattern made from numbers raised to the third power. For the sequence 3, 10, 29, 66, 127:

3 10 29 66 127
7 19 37 61
12 18 24
6 6

Try the formula n^3, and tweak it until you get the right pattern: $n^3 + 2$ in this case. Or if you only need the next number, start at the very bottom and add 6. Then you see that the number 30 goes into the row above that, then 91, and finally 218:

3 10 29 66 127 218
7 19 37 61 91
12 18 24 30
6 6 6

4. Exponential Patterns

Some number sequences increase really, really fast. They increase **exponentially**. An example of an exponential pattern is to keep multiplying by 5:

1, 5, 125, 625, …

No matter how many rows of differences you look at, the change never becomes constant. Fortunately these patterns are usually fairly easy to spot just because the numbers are going up so quickly. The slowest exponential pattern is to keep multiplying by 2:

1, 2, 4, 8, 16, …

That seems to be going up slowly, but the numbers do get really big before you get very far. Just look up the story of the grains of rice and the chessboard. if you place 1 grain on the first square of the chessboard, 2 grains on the next, 4 grains on the next, etc., you might think that you can just keep doing that. But by the time you get to the last square, that would be 18,446,744,073,709,551,615 grains, which is about 636,094,623,231,363 pounds of rice! There is simply not enough rice in the world.

Variables and Constants

If we write an **expression** like n + 3, that expression will have a value that depends on what number we pick for n. The unknown n is a variable, because we can select different values for it. If we choose n to be 5, then n + 3 = 8. On the other hand, n could be 0, or 2, or 30, which would all give different values for the expression n + 3. **The letter n is often used for whole number variables**. If the unknown could also be a decimal number like 4.5, or a fraction like $\frac{3}{7}$, you will usually see a different letter, like x. Sometimes you will be given an expression and asked to evaluate that expression for a given value of the unknown. For example, if you are asked to evaluate $\frac{x}{2}$ for x = 10 you will be expected to come up with an answer of 5. The expression $\frac{x}{2}$ means x divided by 2, and it will always return a value that is ½ of x.

Be very careful when you insert a negative number into an expression. For example, evaluate the expression $x^2 + x$ for x = −6. If I just stick that in, I get $-6^2 + -6 = -36 - 6 = -42$, because the exponent comes first before the minus sign (which is a really a multiplication by −1). However, you may remember from the last chapter that if you square −6 you should get −6 x −6 = 36. To avoid awkward order of operation errors like this, **make a habit of protecting a negative number with parentheses before using it for an unknown**: $(-6)^2 + -6 = 36 - 6 = 30$. In fact, many teachers recommend that you always use parentheses when substituting a number for a variable, so you don't have to think about whether it is positive or negative.

For an **equation** like $x + 4 = 10$, x is also considered to be a variable. You can select any value for x you wish, but only one of those values will make the equation true. Sometimes equations are complicated, and there may be several different values of the unknown that will make the equation true. For example, $x^2 + x = 6$ is true for $x = 2$: $2^2 + 2 = 6$. However, it is also true for $x = -3$. $(-3)^2 + -3 = 6$. There are even some equations that are true for every value of x, such as $1 + x - 1 = x - x + x$.

In the equation $x + 3 = y$, 3 is a constant. It never changes. Sometimes a constant is unknown, or it has a value that is difficult to write like maybe 38.3974×10^{-5}. In these cases you may see a letter in a formula that represents a constant. For example, in Einstein's famous formula $E = mc^2$ the letter c stands for the speed of light, and we don't expect that to change anytime soon. Letters at the beginning of the alphabet like a, b and c are often used for constants, while letters at the end like x, y and z are usually chosen to represent variables.

Portfolio Chapter 3

Write your own recipe formula for something, like how to find your height in inches, or maybe how to find your height in inches when wearing shoes.

Create a pattern and write a formula for it using the letter n. Show that your formula creates the pattern when you substitute 1, 2, 3 and 4 for n.

Chapter 3 Quiz

1. $n - n =$

2. $n + n - n =$

3. $m \times 1 =$

4. $\frac{m}{m} =$

5. Create a formula for the area of a square. Do not use more than two different letters to represent unknowns. The first letter is A to represent area: A = ______

6. Create a formula for the perimeter of a square.

7. The formula for the volume of a box is $V = \ell \times w \times h$, where V is the volume, ℓ is the length, w is the width, and h is the height. Create a formula for the volume of a cube, using only two different letters to represent unknowns.

8. Find a formula for the surface area of a cube.

9. **a)** If I can get 1 cow by trading 4 goats for it, how many goats do I need if I want 5 cows?

b) How many goats will I need for c cows?

10. There are 10 millimeters in a centimeter.

a) How many millimeters are in 8 centimeters?

b) How many millimeters are in c centimeters?

11. A taxi service charges an initial fee of $5.25 plus $2.00 per mile. There is also a charge of 50 cents for each minute the taxi spends idling in traffic.

a) How much will you pay for a 15 mile trip if the taxi waits 10 minutes at traffic lights?

b) What is the cost for a trip of m miles, with w minutes of time waiting at lights?

12. Find a formula that generates the following pattern: 5, 7, 9, 11, 13, …

Don't forget to update your portfolio if you learned something in the quiz.

Chapter 4: Working with Unknowns

Multiplying with Unknowns

Multiplication is closely related to making a profit and getting richer. Therefore, most people have a good understanding of the idea behind the numbers. For example, one $100 bill is good; three $100 bills are better: one $100 bill + one $100 bill + one $100 bill = three $100 bills = $300.

But what if we have an unknown amount? For example, Laura can make some money by selling fruit baskets at a neighborhood market. She hasn't decided yet how much to charge for each basket. As we saw in a previous lesson, x is a popular letter to represent an unknown amount. Let's say the fruit baskets will sell for x dollars each. If Laura sells 3 fruit baskets on Monday, she will take home x dollars + x dollars + x dollars which equals 3 times x dollars or 3 x x ?? Now what idiot decided to use the one letter of the alphabet that also serves as a multiplication symbol? Actually, it was more of a slow accident, like two people backing out of their parking spaces on opposite sides of the driving lane at the same time. The multiplication symbol x was chosen by William Oughtred in England around 1630. At just about the same time in France, Renee Descartes selected x, y and z to use for his variables, with x naturally being the first choice. The accident occurred in extreme slow motion, as books written by both men became popular in their respective countries and their notations were adopted.

Anyway, x is now used up for variables in algebra, so the multiplication symbol has become a dot, as in $3 \cdot x$. Many times this dot can be omitted, since $3 \cdot x$ can be written as 3x without causing confusion. It still means x + x + x or 3 times x, just as 3 apples means 3 times an apple. It is a custom to put the number before the letter so write 3x rather than x3.

$$4 + 4 + 4 = 3 \cdot 4 = 12$$

$$x + x + x = 3 \cdot x = 3x$$

Because people are inherently lazy, it is also a custom not to write the number 1 if you have 1x:

$1x = x$

If there is more than one unknown in your multiplication, place the letters in alphabetical order. So x times y is xy and y times x is also xy.

$a \cdot b = ab$

$b \cdot a = ab$

Writing the letters in alphabetical order allows you to quickly see that you are dealing with the same result. That result, ab, is just a number, and it happens to be the same number as ba. When you need to add $(a \cdot b) + (b \cdot a)$, you can see that this is the same as $ab + ab$, which is 2 times the number ab, or 2ab.

$4 \cdot 5 + 5 \cdot 4 = 20 + 20 = 2 \cdot 20 = 40$

$a \cdot b + b \cdot a = ab + ab = 2 \cdot ab = 2ab$

If you are multiplying an unknown by itself, as in $x \cdot x$, use an exponent instead of writing xx. So, $x \cdot x = x^2$ and $x \cdot x \cdot x = x^3$.

$5 \cdot 5 = 5^2 = 25$

$g \cdot g = g^2$

Because you have been reading for many years, you have no trouble noticing the difference between "The yellow bus" and "The yellow bug". There is also a very big difference between 2x and x^2. 2x means 2 times x, which is the same as $x + x$. x^2 on the other hand means x times x. The same thing goes for 3x and x^3: $3x = x + x + x$, and $x^3 = x \cdot x \cdot x$. If x happens to be 5, 3x is 3 times 5, or 15. Meanwhile x^3 is 5 times 5 times 5, or 125. Read your algebra carefully, and think about what the notation means.

An unknown multiplied by a fraction is shown the same way as an unknown multiplied by a regular number. The number comes first, and then the unknown. The multiplication sign is omitted.

$n \cdot \frac{1}{4} = \frac{1}{4}n$

When you multiply an unknown by a fraction, use the same rules you would use for a regular number.

$5 \times \frac{1}{4} = \frac{5}{1} \times \frac{1}{4} = \frac{5}{4}$

$n \cdot \frac{1}{4} = \frac{n}{1} \cdot \frac{1}{4} = \frac{1n}{4} = \frac{n}{4}$

If you were watching closely just now, you would have noticed that $\frac{1}{4}n$ and $\frac{n}{4}$ mean the same thing. That makes sense, because ¼ of n is the same as n divided by 4.

$\frac{1}{4}n = \frac{n}{4}$

In algebra, if no operation is specified, multiplication is the default. 3x means 3 times x. When we pick a number for x, we often still don't use a multiplication sign. Instead you may see the number in parentheses, like this: when x is 5, 3x is 3(5) which equals 15.

Adding with Unknowns

For this exercise, use identical objects such as beads, legos, m&m's, cardboard squares, or clean pennies.

Create 2 groups of 3 objects [6 objects in total], and another 3 groups of 3 objects. When you look at your work, you will see that you now have 5 groups of 3. Next, create 2 groups of 4 objects, and another 3 groups of 4 objects. Now you should have 5 groups of 4. This is not really surprising; it's just how things work. However, you can extend this idea to groups of an unknown number. Taking 2 groups of "x" objects and 3 groups of "x" objects makes 5 groups of "x" objects. This tells us that $2x + 3x = 5x$.

Now use your objects in the same way to figure out how to add $2x^2 + 3x^2$. Is it $5x^2$, or $5x^4$, or can it be added at all in some consistent way?

How would we add $2x^3 + 3x^3$?

Try to work this out by yourself even if it seems hard, because you will remember it better. Oddly enough it is the effort that counts rather than the result. If it still doesn't make sense after you've given it a good try, you can continue on to the next section.

You have just done some adding with basic unknowns, like $2x + x = 3x$. We can easily replace x with various real numbers to see that this always works.

We have also seen that x^2 means $x \cdot x$, and that there is a very big difference between x^2 and 2x. Sometimes we have to work with both these kinds of quantities, and we need to know what we can add together.

For example, once you have selected some random value for the unknown x, you can see that x^2 is just a number. $x^2 + x^2$ can be added together to make $2x^2$, just as 9 + 9 = 2 times 9. And $2x^2 + 3x^2 = 5x^2$, just like 2 groups of 9 plus 3 groups of 9 is the same as 5 groups of 9.

The separate parts of an equation or expression are called **terms**. Whenever two terms with x have the same power, they can be added:

$2x^3 + 3x^3 = 5x^3$

But what if the powers are different, as in $2x^2 + x$? Can we add that together into 3 of something? Whenever you find yourself wondering about something like this, try it out with real numbers to make things clear. If we choose 3 for x, the problem reads: 2 groups of 9, plus 1 group of 3. If we choose 5 for x, we get 2 groups of 25 plus 1 group of 5. These groups are different sizes, so there is no way to turn them into 3 groups of something. Sometimes students wonder if maybe we could add $2x^2 + x$ into $2x^3$, but if you use real numbers you can see that this really doesn't work. $2x^2 + x$ means $2 \cdot x \cdot x + x$, while $2x^3$ means $2 \cdot x \cdot x \cdot x$.

Mathematicians have long resigned themselves to the fact that $2x^2 + x$ just sits there. However, we can add two expressions like that together. Teachers tell their students to "add like terms":

$$2x^2 + x + 3x^2 + 4x = 5x^2 + 5x$$

Always remember that algebra is a real thing that has to work in real life, **so use a real number to check your conclusions**. Hmm, that last part is a different color. Could it be important? **Yes it is!** In fact, this is really the key to understanding algebra. If it is true that $2x^2 + x + 3x^2 + 4x = 5x^2 + 5x$, then it has to work for any number x. Replacing x with the number 10, we get: $2 \cdot 10^2 + 10 + 3 \cdot 10^2 + 4 \cdot 10 = 5 \cdot 10^2 + 5 \cdot 10$, which says that $200 + 10 + 300 + 40 = 500 + 50$.

When you insert a real number and see that the answer is correct, it is still possible that you have made a mistake somewhere. The answer must be correct for any number x, so if it is really important you may want to try several numbers. **Do not use 1 or 2 as sample numbers.** Although it is tempting to use these small numbers because they are easy to work with, the problem is that they can hide mistakes. 2 + 2 is the same as 2 times 2, so if you've added instead of multiplied, you'll never catch your mistake. The same way, $1^2 = 1$, so if you've forgotten to write ^{2} somewhere you won't notice. If you have more than one unknown you should use a different sample number for each one. My favorite sample numbers are 3, 4, 5, and 10.

A problem you may run into with adding unknowns is the presence of parentheses. Suppose you have to add $x + (5 + x)$. As we saw before, the things inside the parentheses sometimes cannot be done first. Now we have to think of this problem as x + "the whole thing inside the parentheses". To accomplish that we must take x, add 5, and also add x. That gives us $x + 5 + x = 2x + 5$.

And what happens if we try to do $x + (5 - x)$? Here again we must add everything inside the parentheses to x. We add 5, and then we add $-x$. Now we get

$x + 5 + -x = x + 5 - x = 5$.

Both these cases involve adding the things inside the parentheses. As you can see, we can safely remove the parentheses and still solve the problem correctly. This makes it easy to add an expression that is inside parentheses. However once we start subtracting we have to be much more careful.

Subtracting With Unknowns

If you believe that 2x + 3x = 5x, you probably won't be too surprised when I claim that 5x – 3x = 2x. You can easily check that by picking a number for x, like maybe 4. Five groups of 4, minus 3 groups of 4, should be 2 groups of 4. And yes, 20 – 12 is actually 8. Subtraction with unknowns is as straightforward as addition, unless there are parentheses to deal with.

When we reviewed Order of Operations, we looked at the problem 10 – (3 + 2). Because there are no unknowns here, we can just say that 10 – (3 + 2) is the same as 10 – 5. If we think of this problem as 10 minus "the whole thing inside the parentheses", we see that we need to subtract 3, and also subtract 2. We can rewrite 10 – (3 + 2) as 10 – 3 – 2, and still get the same answer. If there are unknowns inside the parentheses we would do the same thing. 10 – (3 + x) can be written as 10 – 3 – x, which is 7 – x. In order to safely remove the parentheses, we have to change the original + sign in front of the x to a – sign. Look at this example carefully, because it is easy to make a mistake here.

10 – (3 + x) = 10 – 3 – x

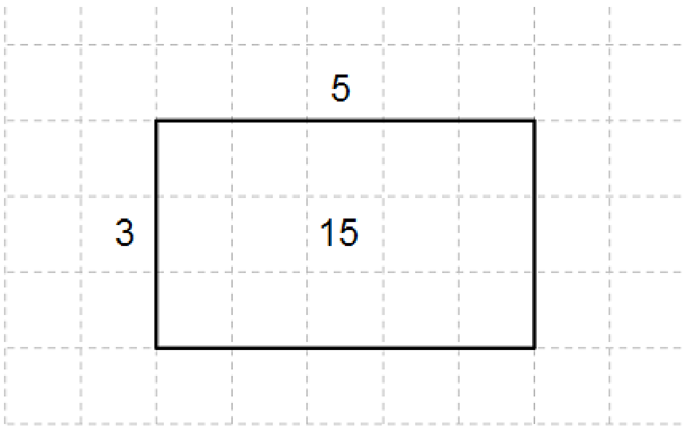
In the same way, we can solve 10 – (3 – x). First we subtract the 3, and then we subtract – x: 10 – 3 – – x = 10 – 3 + x = 7 + x.

10 – (3 – x) = 10 – 3 + x

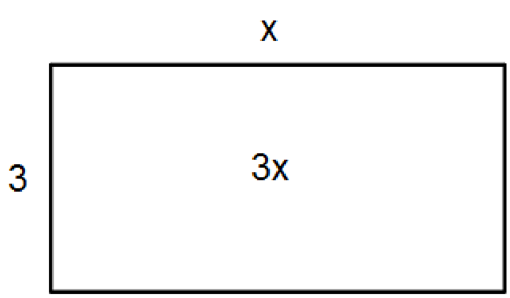
Practice subtracting with an unknown by solving 2x – (6 – x). Use a sample number for x to check your answer!

The Distributive Property

One way to look at multiplication is by considering area. A rectangle with a width of 3 inches and a length of 5 inches has an area of 15 square inches. We can see this clearly by using a grid:

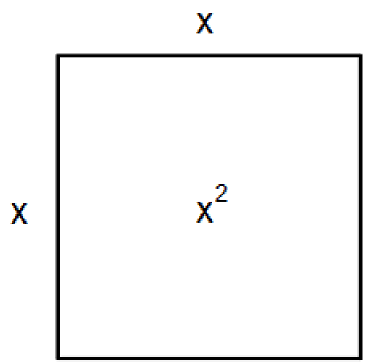


Three squares fit along the width, and 5 squares fit along the length, creating 3 rows of 5 squares. Even without the grid we know that the area of a rectangle with a width of 3 units and a length of 4 units is 15. If the length is increased to 6, the area would be 18. When the length is unknown we can call it x, and still say something sensible about the area:



Here I have selected a random length for x. The area is of this rectangle is 3x. That means that if x is 5, the area is 15. If x is 10, the area will be 30.

The next picture shows a square with unknown sides:

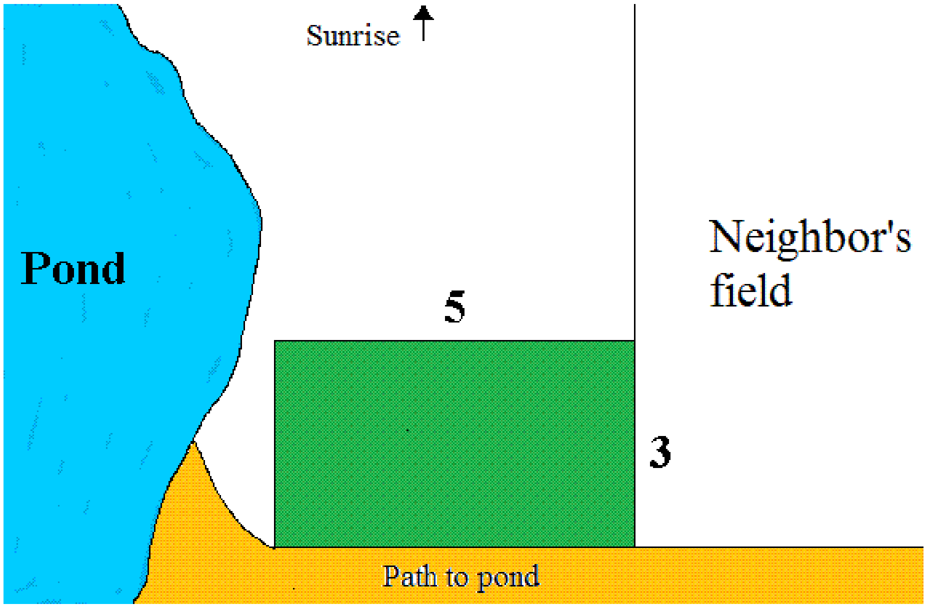


Sometimes it is helpful to represent multiplications with unknowns as squares and rectangles, so you can see what is going on rather than trying to understand it in an abstract way. This is what we will do to help us understand the distributive property, which involves multiplication and parentheses.

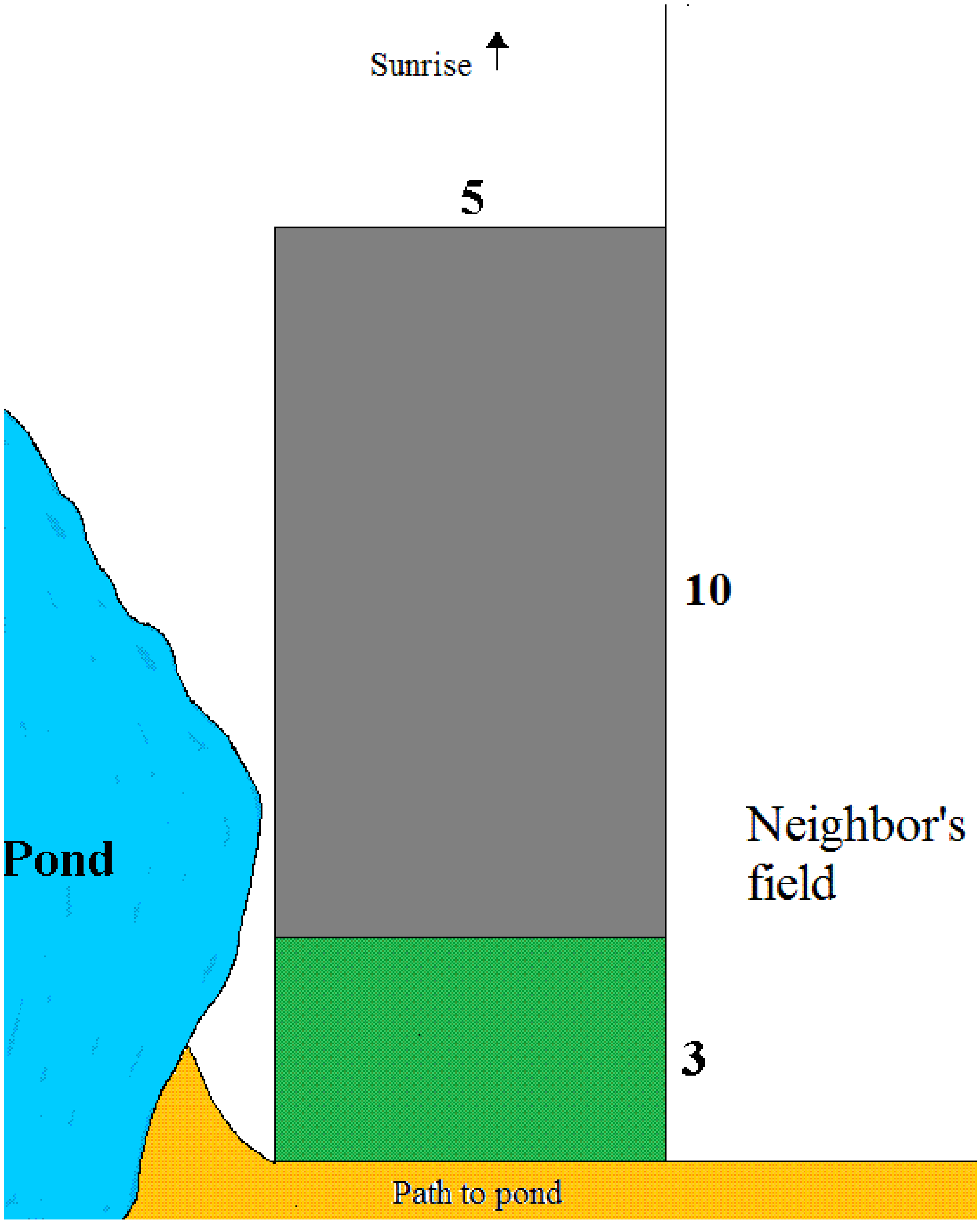
In arithmetic we have no problem doing 5 x (3 + 4). We simply add 3 and 4, then multiply by 5: 5 x (3 + 4) = 5 x 7 = 35. In algebra, however, you will be faced with problems like this: 5(3 + r) = ?. [Notice that the multiplication symbol is gone. Wherever there is no symbol you just have to remember that you are supposed to multiply]. 3 + r should be done first, but how can we add these quantities when we don't know what r is? Now we might start thinking of the problem as 5 times (the whole thing inside the parentheses that we can't add). But how do we do that? Actually, this problem has been solved before, so let's see how other people did it.

For the original solution we have to travel far back in time, before the great pyramid was built......

And here we are, on a Mesopotamian farm. Farmer Hakkem has been thinking about expanding the small back field near the pond. This would give him a bigger crop, but it would also cost money. A bigger field needs more seeds for planting, more labor for plowing and harvesting, and most importantly, there will be more tax to pay. Yes, even in those days there were taxes, and they could be rather high depending on the lifestyle of the local ruler. Here is a picture of the back field (the green area), measured in rods. The rod is an ancient unit of measurement that is about 20 feet long.



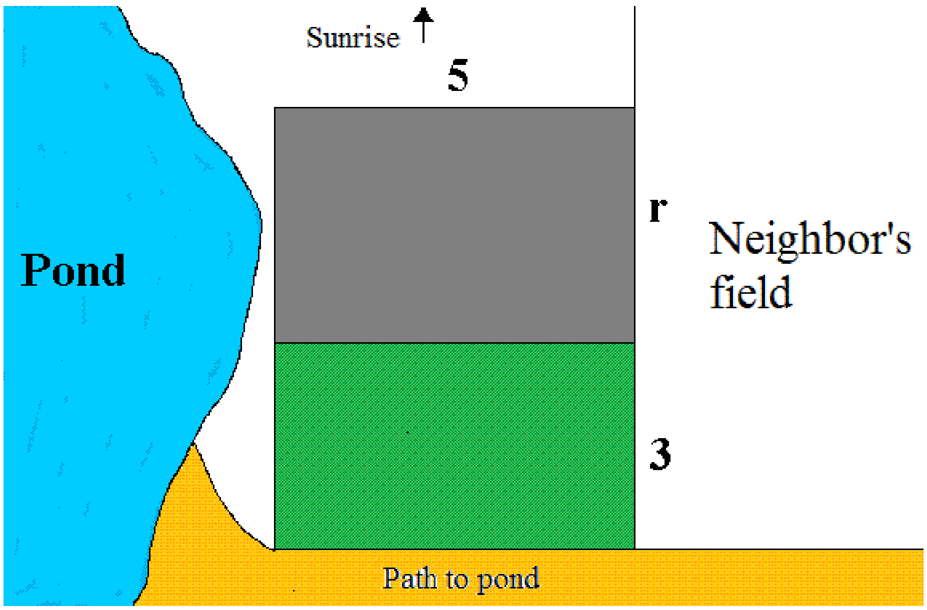
You should be able to see that the current area of the field is 3 times 5 or 15 square rods. If you look really closely, you can even see where the ox had an accident while pulling the plow to prepare the field for planting. There is only one way to expand the field, so let's add another 10 rods in the direction of the sunrise:



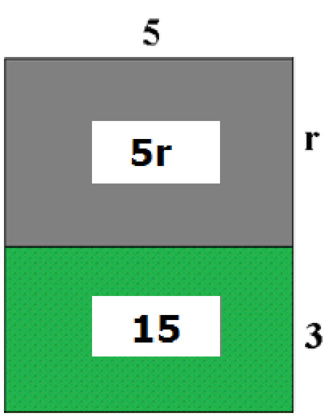
Hakkem can see that the new size of the field is the width times the length, which is 5 times (3 + 10), or 5 times 13 which is 65. So, 5 · 13 = 5 times 3 [the green area] plus 5 times 10 [the grey area]. Unfortunately that turns out to be a little more than Hakkem can afford. However, he notices that 5(3 + 10) is the same as (5 times 3) + (5 times 10).

$5(3 + 10) = 5 \cdot 3 + 5 \cdot 10$

Hakkem doesn't know how many rods he should expand the field instead, so he calls this unknown distance r. We can imagine it like this:



Again Hakkem can see that the total area of the field is the length times the width, which is 5 times (3 + r). Looking at his last drawing, he reasons that 5(3 + r) must be 5 times 3 [the green area] plus 5 times r [the grey area].



$5(3 + r) = 15 + 5r$

This gives farmer Hakkem has a simple way to calculate his new taxes, since the additional area is 5r square rods. For example, if he chooses r to be 4, that gives him an additional 20 square rods to pay taxes on.

This concept is called "**the distributive property**". It is the same thing you use when you need to calculate something like 3 times 14. The easiest way to do that without a calculator is to multiply 3 times 10 first and then add 3 times 4:

$$3 \cdot 14 = 3(10 + 4) = 3 \cdot 10 + 3 \cdot 4 = 42$$

Now, I know what you're going to say: all we did here was to change $5(3 + r)$ into $15 + 5r$. Why is that so much better? The important thing though is that we were able to change it. This change allows mathematicians to take complex equations and simplify them. For example, calculus would have been impossible to develop without the distributive property. Using methods like this scientists make important discoveries about the universe, because they can work around unknowns and continue learning.

Negative Numbers and the Distributive Property

Negative numbers distribute out just like positive numbers:

$$-4(x - 7) = -4 \cdot x + -4 \cdot -7$$

$$-4(x - 7) = -4x + 28$$

Sometimes the distributive property seems a bit confusing when the negative number is part of a subtraction, as in $6x - 5(x + 4)$. Students often wonder if the minus sign is "part of the 5", or if they should just multiply by positive 5. Well, if you go by the order of operations, you should multiply first, and then do the subtraction. The problem says to take 6x, and then subtract $5(x + 4)$. Unfortunately that is easier said than done. $5(x + 4)$ is $5x + 20$, and all of that has to be subtracted from 6x. That means you would have to use extra parentheses:

$$6x - 5(x + 4) = 6x - (5(x + 4))$$

$$6x - 5(x + 4) = 6x - (5x + 20)$$

Now make sure to subtract 5x and also subtract 20:

$6x - 5(x + 4) = 6x - 5x - 20$

$6x - 5(x + 4) = x - 20$

That seemed like a lot of work. In practice, people prefer to make the minus sign part of the 5, and they do it without thinking. Here is how it really works:

$6x - 5(x + 4) = 6x + -5(x + 4)$

$6x - 5(x + 4) = 6x + -5x - 20$

$6x - 5(x + 4) = 6x - 5x - 20$

Because this kind of situation is so common in algebra, it is perfectly acceptable to attach the minus sign to the 5 without placing an extra plus sign, just like this:

$6x - 5(x + 4) = 6x - 5x - 20$

Practice

There are many websites that offer practice problems for the distributive property. Do at least 10 problems in a row correctly before you decide that you know how it works.

Portfolio Chapter 4

Now is a good time to ask a "what if" question. What if there were more terms, like $x(5x + 2 + a)$? Does the distributive property still work? Why or why not?

For this chapter you could create a picture with rectangles that shows how you can multiply $5x + 2 + a$ by x. On the other hand, you could also use real numbers to show whether or not you can use the distributive property in this situation. To show that it doesn't work you need only one example. To show that it does, use a minimum of two examples with different numbers. Do not use 1 or 2 as sample numbers.

Chapter 4 Quiz

1. You learned that $3a = a + a + a$. What is $3a + 5a$? Check your answer using a specific number value for a.

2. $12b - 10b =$

3. $b + b + b =$

4. $ab + ab + ab =$ [To check your answer pick a number for a and a number for b.]

5. $x - 0.5x =$

6. A laptop that normally costs $x is on sale at 10% off. How much does it cost now? Express your answer in terms of x.

7. The inhabitants of Vrinn use a lot of bloogs. Because bloogs are fairly small, they are bought and sold by the splork rather than individually. Assuming there are b bloogs in a splork, how many bloogs are there in 5 splorks? [If you have difficulty with this question, please postpone interplanetary travel until your algebra skills improve.]

8. y^2 means:

a) 2y

b) y2

c) $y \cdot y$

d) $y + y$

9. $2 \cdot 4a =$

10. $10a \cdot -5 =$

11. $7b \cdot 3 =$

12. $-3 \cdot -3a =$

13. $x^2 + x^2 =$

14. $x^2 \cdot x^2 =$

15. $2x^2 + x + x^2 =$

16. Mark's grandfather is three times as old as Mark. If Mark is y years old, his grandfather is______ years old.

For the next 6 questions, use the distributive property. Check your answers using real numbers.

17. $5(8 + a) =$

18. $12(f + 5) =$

19. $3(4 - x) =$

20. $-6(y - 4) =$

21. $a(b + c) =$

22. $-a(b - c) =$

23. $12x - 22 - 10(x - 2) =$

24. I draw a rectangle with width x and length y. Next I want to make a new rectangle that is three times as big, so I use 3x as the width and 3y as the length. The AREA of my new rectangle is:

a. 3 times as big as the area of the original rectangle

b. 6 times as big as the area of the original rectangle

c. 9 times as big as the area of the original rectangle

d. 12 times as big as the area of the original rectangle

25. True or False: $3x - 3 = x$

26. True or False: $5x^2 - x^2 = 5$

Chapter 5: Division

Working with Fractions

Make sure that you really understand how to work with fractions. Here are the basics:

1. A fraction is really a division. $\frac{1}{4}$ means 1 divided by 4, which is one quarter. We can read $\frac{8}{4}$ as "8 quarters", and simplify it to $\frac{2}{1}$ or 2. However, it also means 8 divided by 4, which is 2.

2. We can draw fractions to see what is happening. Pie fractions are easiest to draw. You should always imagine your favorite kind of pie when you are doing this so that you learn to like fractions. Drawing pies helps you understand that 3 wholes is the same as 12 quarters. When you look at the size of the pie pieces, you can see that $\frac{1}{3}$ is bigger than $\frac{1}{4}$, even though 4 is a larger number than 3.

3. Some fractions are the same size as others, for example $\frac{1}{2}$ is the same size as $\frac{2}{4}$, and $\frac{1}{3}$ is the same size as $\frac{2}{6}$. We can see this easily when we are working with pies. We can change the size of fraction pieces by cutting them up, or mathematically by multiplying the top and bottom by the same number: $\frac{1}{2} = \frac{1 \text{ x } 4}{2 \text{ x } 4} = \frac{4}{8}$. We can also combine smaller pieces to make bigger fractions, or divide the top and bottom by the same number.

$$\frac{4}{8} = \frac{4 \div 4}{8 \div 4} = \frac{1}{2}$$

4. If you want to add or subtract fractions, the pieces must be the same size. Otherwise it really doesn't make sense to try to add or subtract. $\frac{1}{2} + \frac{1}{4} = ?$ There are two pieces, but they are both different. Change $\frac{1}{2} + \frac{1}{4}$ to read $\frac{2}{4} + \frac{1}{4}$. Now you have 3 pieces, 2 *quarters* and 1 *quarter*. All pieces are the same size so you can add them:

$\frac{1}{2} + \frac{1}{4} = \frac{2}{4} + \frac{1}{4} = \frac{3}{4}$

Many times you will have to change the size of both pieces: $\frac{1}{3} + \frac{1}{4}$ = ? Look for a number that both 3 and 4 "go into". That number is 12. Change both fractions to twelfths: $\frac{1}{3} = \frac{4}{12}$, and $\frac{1}{4} = \frac{3}{12}$. Then you can add $\frac{4}{12}$ and $\frac{3}{12}$, which is $\frac{7}{12}$.

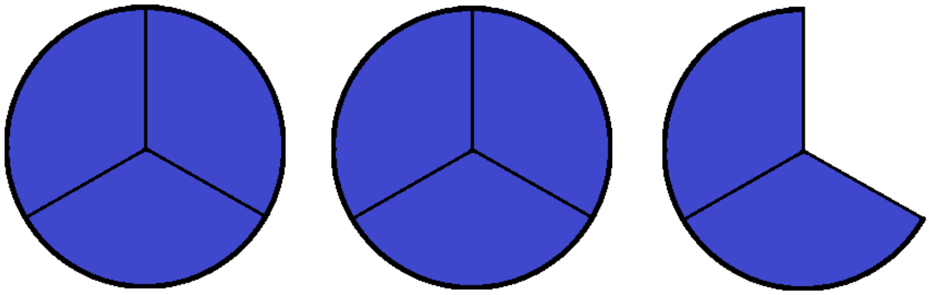
5. To get a fraction of a certain amount, multiply that amount by the fraction. $\frac{1}{4}$ of 200 is $\frac{1}{4} \times \frac{200}{1} = \frac{200}{4}$, which is 50.

6. Multiplying fractions is easy: you just multiply across. $\frac{1}{5} \times \frac{1}{4} = \frac{1 \times 1}{5 \times 4}$, which is $\frac{1}{20}$. If you need to multiply by a whole number, you can turn it into a fraction: $3 = \frac{3}{1}$. On the other hand, you can just think about what it really means. Three times $\frac{1}{5}$ means that you have three of those pieces: $3 \times \frac{1}{5} = \frac{3}{5}$

7. when you divide by a fraction, you multiply by its reciprocal (flip the fraction around):

$6 \div \frac{1}{2} = \frac{6}{1} \div \frac{1}{2} = \frac{6}{1} \times \frac{2}{1} = \frac{12}{1} = 12$

8. A mixed number is composed of a whole number and a fraction, like this: $2\frac{2}{3}$. Mixed numbers can always be converted to fractions so you can follow the rules above. To learn how to do that, just start by drawing pies. For $2\frac{2}{3}$, draw three pies and cut the last one into thirds. Then erase one of the pieces so you have two whole pies and two thirds. Next, cut up your whole pies to see how many thirds you have altogether:



There are 8 thirds, so $2\frac{2}{3} = \frac{8}{3}$. Once you have done that quite a few times, you'll know how it works and you won't have to draw pictures anymore. A fraction that has a larger numerator than its denominator is called an improper fraction. To change such a fraction to a mixed number, take the pieces and make whole pies out of them. If you have $\frac{13}{4}$, you can make 3 whole pies. Each of those pies takes 4 quarters to make, so that uses up 12 pieces. Now you have 1 quarter piece left over: $\frac{13}{4} = 3\frac{1}{4}$.

All of the rules work the same way when your fractions contain unknowns. If you know that $\frac{1}{5} + \frac{3}{5} = \frac{4}{5}$, you can figure out that $\frac{x}{5} + \frac{3x}{5} = \frac{4x}{5}$. Try it out in the quiz at the end of this chapter.

Dividing with Unknowns

Pay close attention while I do a magic trick. First, you think of a number, any number between 1 and 100. Got it? Now multiply your number by 5. Next take the resulting number, and divide it by 5. Okay, now tell me the result, and I'll tell you what your original number was! What, you're not impressed? Well, let's face it; algebra is not magic. The very nature of algebra is that it is completely predictable, but that is exactly what makes it both amazing and useful. If we take a number and multiply it by 5, and then divide by 5, we always get our original number back. Using n for "a number", we can write:

$$\frac{5n}{5} = n$$

[Remember that the little line in the middle of a fraction is a division line, so this says 5n ÷ 5 = n].

Of course this is not limited to the number 5; any number will do the same thing in this case. We can multiply and divide by 4, or by 10, or.... Using a instead of 5, we get

$$\frac{an}{a} = n$$

Now you can see that we are actually getting somewhere with our algebra. The part on the right of the equals sign definitely looks less complicated than the part on the left.

As a general rule, we say that we can remove, or cross off, numbers above and below the division line if they are the same. To illustrate this further, let me do another really lame trick. Think of a number between 1 and 20. Since I don't know this number, I'll call it x. Now multiply your number by 3, to get 3x. Next, divide the result by your original number x. The number you end up with is...3 [boos from audience]. The trick looks like this:

$$\frac{3\cancel{x}}{\cancel{x}} = 3$$

$$\frac{12}{3} = \frac{\cancel{3} \cdot 4}{\cancel{3}} = 4$$

$$\frac{3x}{3} = \frac{\cancel{3} \cdot x}{\cancel{3}} = x$$

$$\frac{3x}{x} = \frac{3 \cdot \cancel{x}}{\cancel{x}} = 3$$

When you take an expression like $\frac{an}{a}$, and remove the a's from it, what you are really doing is replacing the a's with 1's.

$$\frac{a}{a} = \frac{1a}{1a} = \frac{1}{1}$$

$$\frac{an}{a} = \frac{1n}{1} = n$$

an ÷ a = 1n ÷ 1 , which is n.

DO NOT replace the numbers or letters you remove above and below the division line with zeros. This becomes important when there is nothing left above the line. For example, simplify $\frac{b}{4b}$. This means you are taking $\frac{1}{4}$ and multiplying by b, then dividing by b, which is redundant. Remove the b's to get $\frac{1}{4}$, NOT $\frac{0}{4}$ which is zero and would leave you with nothing at all!

$$\frac{b}{4\,b} = \frac{1\,b}{4\,b} = \frac{1\,\cancel{b}}{4\,\cancel{b}} = \frac{1}{4}$$

Five divided by 2 can be written like this: $\frac{5}{2}$. In the same way, we can divide any number by 2: $\frac{n}{2}$. This means "the number n divided by 2". In the last chapter we saw that $\frac{n}{2}$ is the same as $\frac{1}{2}$ times n, which multiplies as $\frac{1}{2} \cdot \frac{n}{1} = \frac{1n}{2} = \frac{n}{2}$.

$$\frac{n}{2} = \frac{1}{2}n$$

Now, suppose I have half of an unknown number, like maybe $\frac{1}{2}$ x, and I'd like to have a whole x instead. To change $\frac{1}{2}$ x into x, I would have to multiply by 2:

$$2 \cdot \frac{1}{2} x = \frac{2}{1} \cdot \frac{1}{2} \cdot x = x$$

That makes sense, because 2 times half an apple would be a whole apple. Also, the 2 on the bottom cancels out when you multiply by two, which might be easier to see if you write it like this:

$$2 \cdot \frac{x}{2} = x$$

There are many occasions in algebra when we can look at an expression, which is a bunch of numbers and letters without an equals sign, and eliminate something from it. Consider this expression: $\frac{10x(8 + y)}{5x}$.

This may look scary, but we can quickly cut it down to size. If it is not obvious at first that we are doing some multiplying and dividing by the same numbers, just re–write it like this:

$$\frac{2 \cdot 5 \cdot x \cdot (8 + y)}{5 \cdot x}.$$

You should now be able to see that the 5 and the x are doing nothing useful here, so we can get rid of them:

$$\frac{2 \cdot \cancel{5} \cdot \cancel{x} \cdot (8 + y)}{\cancel{5x}}$$

Now we have $2(8 + y)$, which is a lot simpler.

Next, we could use the distributive property to change $2(8 + y)$ into $16 + 2y$.

On the other hand, we could use the distributive property first, before we've done any dividing.

$$\frac{10x(8+y)}{5x} = \frac{80x + 10xy}{5x}$$

While the second expression looks different, it is equal to the first. So, if we take this second expression and simplify it, it should also work out to 16 + 2y.

Let's give it a try: $\dfrac{80x + 10xy}{5x} = \dfrac{16 \cdot 5 \cdot x + 2 \cdot 5 \cdot x \cdot y}{5 \cdot x}$

Notice that on each side of the equals sign there are two things that have to be divided by 5x. We call these things "terms". Terms are separated by either a + or a − sign. There are two terms above the division line, and each term individually needs to be divided by 5x. Divide 80x by 5x, and also divide 10xy by 5x:

$$\frac{80x + 10xy}{5x} = \frac{80x}{5x} + \frac{10xy}{5x}$$

$\dfrac{80x}{5x} = 16$ and $\dfrac{10xy}{5x} = 2y$, so our answer is $16 + 2y$ again. In algebra there are often many different ways of doing something, but they always have the same result.

By the way, dividing terms incorrectly can really trip you up, so look at the following examples carefully:

$\dfrac{8b + 8}{2} = 4b + 4$. Both terms are divided by 2 to get the answer.

$\dfrac{6abd}{3} = 2abd$. Since there is only one term, that's all you divide.

$\frac{5c-2}{5} = c - \frac{2}{5}$, **NOT c − 2!**

Conversion Factors

Oddly enough, crossing out things above and below the division line can be done with more than just numbers. It also helps to use this method when you are converting from one unit to another. Normally we don't include units like pounds or dollars in our algebra notation when there is only one type of unit, since it just clutters things up. But when you need to convert between units, you'll want to keep them around and work with them using regular algebra.

Suppose you have a value of 330 minutes, and you need to know how many hours that is. You can use your algebra skills to figure it out. Start by creating a simple equation:

1 hour = 60 minutes

Divide both sides by 60 minutes:

$$\frac{1\text{hour}}{60\text{ minutes}} = \frac{60\text{ minutes}}{60\text{ minutes}}$$

$$\frac{1\text{hour}}{60\text{ minutes}} = 1$$

The fraction $\frac{1\text{hour}}{60\text{ minutes}}$ is called a conversion factor. You can read it as 1 hour per 60 minutes or 1 hour divided by 60 minutes. Note that we could also have divided both sides of the equation by 1 hour, to get 1 = $\frac{60\text{ minutes}}{1\text{ hour}}$. Sixty minutes per hour is also a valid conversion factor, but it is not as useful in this particular situation. The value of a conversion factor is always 1. As you know, you can multiply any number, or even any unknown, by 1 without creating a change. Take the original value of 330 minutes and multiply it by the first conversion factor:

$$330 \text{ minutes } \times \frac{1 \text{hour}}{60 \text{ minutes}}$$

$$\frac{330 \text{ minutes}}{1} \times \frac{1 \text{hour}}{60 \text{ minutes}} = \frac{330 \text{ minutes} \cdot 1 \text{ hour}}{60 \text{ minutes}}$$

Now the unit minutes cancels out:

$$\frac{330 \cdot 1 \text{ hour}}{60} = \frac{330 \text{ hours}}{60} = \frac{330}{60} \text{ hours} = 5.5 \text{ hours}$$

If you arrange your equation so that you are both multiplying and dividing by the unit you want to get rid of, it will magically disappear. When you actually go to do this yourself, you might feel slightly confused if you accidentally use the wrong conversion factor:

$$330 \text{ minutes} \times \frac{60 \text{ minutes}}{1 \text{ hour}} = ??$$

Oops, that doesn't look right. We don't have something above and below the division line that we can cross off. In fact, we end up with a mess of 19800 min^2/hr. This problem can easily be fixed by turning the conversion factor the other way around. Just as there are 60 minutes per hour, there is also 1 hour per 60 minutes, so that is the one you want:

$$330 \text{ minutes} \times \frac{1 \text{ hour}}{60 \text{ minutes}} = \frac{330 \cancel{\text{minutes}} \times 1 \text{ hour}}{60 \cancel{\text{minutes}}} = 5.5 \text{ hours}$$

You can also convert more complex units, such as miles per hour. If a high-speed train is travelling at 240 miles per hour, how fast is that per minute? Well, we can just use the conversion factor:

$$\frac{240 \text{ miles}}{1 \text{ hour}} \times \frac{1 \text{hour}}{60 \text{ minutes}} = \frac{240 \text{ miles}}{60 \text{ minutes}} = \frac{4 \text{ miles}}{1 \text{ minute}}$$

So, the train is covering a distance of 4 miles in just 1 minute, and its speed is 4 miles/minute.

The metric system is very important in science, and you will need to know how to convert familiar measures like inches and pounds to metric units like centimeters and kilograms. To do so you will need the proper conversion factors, which are always readily available online.

The following video explains metric units and shows you how to make conversions. Spending some extra time on this topic will make your high school chemistry course much easier.

SI Units and Metric Conversions - YouTube

Quiz: Let's Work Together – It'll Go Faster

In this section, we will discuss a certain type of problem that is often found on tests. It can look confusing, but does not require a lot of algebra skills to solve. Use conversion factors as needed.

Question 1

Paul can wash 2 cars an hour. Mike can wash 3 cars an hour. When Paul and Mike work together, how many cars can they wash per hour? No, this is not a trick question; it really is not very hard. _______ cars

Question 2

Paul can wash 2 cars an hour. Mike can wash 3 cars an hour. They can wash 5 cars an hour when they work together. How many minutes does it take them to wash one car when they work together? _______ minutes

Question 3

Paul can wash a car in 30 minutes. Mike can wash a car in 20 minutes. How many minutes will it take them to wash a car when they work together?

Isn't it amazing how the same question can now be so much more complicated? Previously, the question gave the rate at which Paul and Mike washed cars. We can add these two rates up to find how fast Paul and Mike can work together. We can't add 30 minutes and 20 minutes here to come up with 50 minutes; that wouldn't make sense. When the problem doesn't give you the rate directly, you must find it yourself. Try doing that for this problem

Paul's hourly rate: ____ cars per hour + Mike's hourly rate ____ cars per hour = ____ cars per hour. This means 1 car takes _____ minutes when Paul and Mike work together.

Another way to express the rate is directly as a fraction rather than hourly:

Paul's rate: $\frac{1\text{ car}}{30\text{ minutes}}$ + Mike's rate: $\frac{1\text{ car}}{20\text{ minutes}}$

Now add the two fractions to get the combined rate. Since the units are the same for both fractions we can leave them out of our calculations. The answer will be in the same units, cars per minute:

$$\frac{1}{30} + \frac{1}{20} = \frac{2}{60} + \frac{3}{60} = \frac{5}{60} = \frac{1}{12}$$

Question 4

Although this type of question usually involves washing cars, sometimes you'll see one with machines working at different rates (older and newer equipment I guess).

Bob has two copy machines. The first copier can complete a certain copy job in 5 minutes, while the second would take 20 minutes to produce that number of copies. How long will the job take when both machines work together?

Remember that the only thing you can add is the rates. The first machine is working at a rate of 1 job per 5 minutes, and the second machine is working at a rate of 1 job per 20 minutes.

Rearranging Equations

If $\frac{6}{2}$ = 3, that means that $\frac{6}{3}$ = 2, and 2 times 3 = 6. That may seem very basic, but make sure you really truly understand it.

$\frac{20}{5}$ = 4, so 20 = 4 · 5. In general: $\frac{a}{b} = c$, so $a = b \cdot c$

$\frac{20}{5}$ = 4, so $\frac{20}{4}$ = 5. In general: $\frac{a}{b} = c$ so $\frac{a}{c} = b$

There will be many occasions where you need to rearrange something like $\frac{pq}{r}$ = s, and you should immediately be ready to say that pq = rs or that $\frac{pq}{s}$ = r. Feeling comfortable with this saves a lot of time, which can be very helpful on a college admissions test.

Here is how it works in practical terms. Suppose Kevin wants to get an A (90% or higher) in algebra for the current grading period, which involves taking three tests in addition to a homework grade. All four grades will be of equal value. So far he has taken 2 tests, and received grades of 79 and 86. His homework grade is 100%. What is the minimum grade he needs to get on his final exam?

Although Kevin could use arithmetic here, algebra will provide the answer quickly. He calls the unknown grade g, and calculates his desired average like this:

$$\frac{79 + 86 + 100 + g}{4} = 90$$

There is a lot of stuff being divided by 4, but it still is some kind of number:

$$\frac{\text{a number}}{4} = 90$$

If $\frac{6}{2}$ = 3, then 6 = 2 times 3. If $\frac{\text{a number}}{4}$ = 90, then that number is 4 times 90, which is 360. Now we just have a simple equation:

79 + 86 + 100 + g = 360

265 + g = 360

The value of g is the difference between 360 and 265, and 360 – 265 = 95. Kevin needs to get a grade of at least 95% on his final test to get the A he wants.

Factoring

The distributive property involves multiplication to change 5(3 + a) into 15 + 5a. We can use division to work backwards. This is called **factoring**. If we have 15 + 5a we divide both terms by 5, and then put the 5 outside the parentheses: 15 + 5a = 5(3 + a). Just look for the biggest thing that both terms have in common.

If you have 16p + 4pq, you could put a 4 outside the parentheses like this:

16p + 4pq = 4(4p + pq)

That isn't wrong, but p can also go outside the parentheses:

16p + 4pq = 4p(4 + q)

Needless to say, you should check your work by multiplying that back out, so 4p(4 + q) = 16p + 4pq.

That's nice, but why bother doing this "factoring" in the first place? Well, for one thing, sometimes it can help you solve equations. Suppose we have an equation like $2x^2 + 3x = 0$, and we want to know the value of x. You could spend quite a while guessing, but it is much easier to factor. What both $2x^2$ and 3x have in common is x, so put that outside the parentheses. $2x^2$ divided by x is $\frac{2 \cdot x \cdot x}{x}$. Since you are multiplying by x, and also dividing by x, one x cancels to give you 2x. 3x divided by x is just 3:

$2x^2 + 3x = 0$

$x(2x + 3) = 0$

Now, x is some number, and so is 2x + 3. If you multiply two numbers and the result is zero, then one of those numbers has to be zero, or maybe they are both zero. So, either x is zero, or 2x + 3 is zero. For that last possibility, just solve for x: 2x + 3 = 0, 2x = -3, and x must be $\frac{-3}{2}$. Try x = 0 in the original equation, and you'll see that it works.

So does $-\frac{3}{2}$: $2x^2 + 3x = 0 \rightarrow 2\left(-\frac{3}{2}\right)^2 + 3\left(-\frac{3}{2}\right) = 0 \rightarrow 2\left(\frac{9}{4}\right) - \frac{9}{2} = 0 \rightarrow \frac{18}{4} - \frac{9}{2} = 0$.

Both solutions are correct.

Another use for factoring is to simplify expressions. If you have something complex looking like $\frac{3x}{9x^2 + 6x} = 2$, you can factor the bottom part like this:

$$\frac{3x}{9x^2 + 6x}$$

$$\frac{3x}{3x(3x + 2)}$$

Now 3x cancels to give you a much simpler fraction:

$$\frac{1}{3x + 2}$$

If you are asked to factor something, you can save yourself some trouble by arranging it neatly first. For example, 7x + 9 – 2x + 6 should be rearranged to 7x – 2x + 9 + 6, which is 5x + 15. Now we can quickly see that both terms have 5 as their largest common factor. This expression can be factored into 5(x + 3).

Practice Factoring:

1. 18q + 9z =
2. 8s + 2 – 4s =

3. $24xy + 6x =$
4. $20xy - 4xz =$
5. $16x + 8s + 4 =$

Answers after next section

1. $9(2q + z)$
2. $2(2s + 1)$
3. $6x(4y + 1)$
4. $4x(5y - z)$
5. $4(4x + 2s + 1)$

Odds and Evens

The **integers**, or whole numbers, consist of positive whole numbers, negative whole numbers, and 0. Even integers are those positive or negative numbers that can evenly divided by 2 (with the result being another integer). Odd numbers cannot be divided by 2 in this way. The number zero is even! You can divide it by 2 and get an integer (0).

Adding even or odd numbers gives a predictable result. Complete the following table:

even + even	=	even
even + odd	=	
odd + odd	=	
even times even	=	
even times odd	=	
odd times odd	=	

You probably used some examples to help you fill in the table. But how can you be sure that every set of numbers you try will give the same results? Let's see.

Because all even numbers are divisible by 2 they are usually represented as 2n, where n is an integer. For 2n, if $n = 1$, $2n = 2$ which is even. If $n = -3$ then $2n = -6$ which is also even.

Odd numbers can be represented as $2n - 1$, or $2c - 1$. If $c = 4$, then $2c - 1 = 7$ which is odd.

Because we will be dealing with two different even and odd numbers, we will use a, b, c and d as our variables instead of n. These letters will always represent integers in the examples below.

Now try adding two even numbers:

$2a + 2b =$

You can factor this to get $2(a + b)$, which can be evenly divided by 2 to give $a + b$

Example: $a = 1$ and $b = -2$

$2a = 2$ and $2b = -4$. $2 + -4 = -2$ which is even.

Try using your factoring skills to show that the other answers in the table are always true.

even + even	=	$2a + 2b$	$2(a + b)$	even
even + odd	=	$2a + (2c - 1)$		
odd + odd	=	$(2c - 1) + (2d - 1)$		
even times even	=	$2a \cdot 2b$		
even times odd	=	$2a \cdot (2c - 1)$		
odd times odd	=	$(2c - 1) \cdot (2d - 1)$	See Ch. 14	odd

Factoring Practice Answers:

1. $9(2q + z)$
2. $2(2s + 1)$
3. $6x(4y + 1)$
4. $4x(5y - z)$
5. $4(4x + 2s + 1)$

Summary

You're making a lot of progress at this point. In just 5 chapters you have learned all of the basic tools of algebra. These are the tools that will help you all the way through calculus, so let's review and make sure you have a firm grip on them:

- **Using a letter to replace an unknown quantity**
- **Algebra notation, including omitting the multiplication sign, and using • , ≥, ≤, >, and <**
- **Manipulating negative numbers, and the absolute value sign |x|**
- **Understanding that 3a means a + a + a**
- **Temporarily replacing an unknown with a number so that you can work with it more easily, or check your work.**
- **The distributive property: 5 (a + b) = 5a + 5b**
- **Working backwards from the distributive property (also called factoring)**
- **Understanding that $\frac{3a}{3}$ = a and $\frac{3a}{a}$ = 3**
- **Rearranging $\frac{a}{b}$ = c into $\frac{a}{c}$ = b or a = bc**

These are not "yeah I'll take your word for it" kind of things. They have to be YOUR tools, not my tools, so you have to really thoroughly understand them. Speaking of tools, I was already about 25 years old when I first tried using a saw. No one was around to help me, nor had I ever seen anyone using a saw close up. I tried to move the saw back and forth while forcing it through the wood at the same time. The harder I pushed down, the more the saw became stuck. After much frustration I realized that I would have to concentrate on moving the saw back and forth properly, and just trust that it would eventually cut through the wood as it was designed to do. When I first used algebra tools, it was the same way. In my rush to get the answer I would fail to apply the tool correctly, and get hopelessly stuck in the problem. In the next quiz, you'll be able to practice using your tools. Find the right tool for the problem, and concentrate on using it correctly. Let the tool do the work.

Portfolio Chapter 5

Why is it okay to cross things off if they are located both above and below the fraction line?

Why can you multiply something by a "conversion factor"? Doesn't that change the actual value?

How do you know which unit goes on top of the conversion factor and which one goes on the bottom?

Suppose that AQ = $\frac{nT}{Q}$, and you know that A is 5, Q is 12, and T is 7.2. How would you go about finding n?

Chapter 5 Quiz

[For any division involving unknowns, it is assumed that the value of such unknowns would not result in a division by zero.]

1. $\frac{-10x}{-5x} =$

2. $\frac{-12b}{12} =$

3. $-12a \div -2 =$

4. $10a \div 5 =$

5. $40c \div -4 =$

6. $\frac{d}{d} =$

7. $\frac{-4ac}{2a} =$

8. $\frac{x}{2} + \frac{x}{2} =$

9. Just like I did in the last quiz, I draw a rectangle with width x and length y. Next I want to make a new rectangle that is three times as big, so I use 3x as the width and 3y as the length. The PERIMETER of my new rectangle is:

a. 3 times as big as the original perimeter
b. 6 times as big as the original perimeter
c. 9 times as big as the original perimeter
d. 12 times as big as the original perimeter

(Be careful – add up the sections around the perimeter of both rectangles and compare them. Then use some sample numbers to see if you were right.)

For the next 7 questions, use the distributive property backwards. Divide both terms by *the largest possible amount* and then put that amount outside the parentheses. Use the following format: 4x + 2 = 2(2x + 1).

10. 40 + 5a =

11. 12f + 60 =

12. 12 – 3x =

13. 6y + 24xy =

14. ab + ac =

15. a^2 – ac =

16. $4a^2$ – 4ac =

17. $4x^2 - 4ax + x =$

18. Susan wants to visit her aunt in Miami, 100 miles from where she lives. How long would it take for her to get there if

a. She rides her bike at a speed of 10 miles/hr

b. She drives her mom's car there at an average speed of 50 miles/hr

c. She decides to hitchhike and gets picked up by an alien spaceship which travels at m miles per hour.

19. How do we measure how fast something is going? We look at the distance it covers in a certain amount of time. For example, if a car drives 1 mile in 2 minutes, we would say that its speed is 0.5 miles per minute. Then we could use a conversion factor of 60 minutes per hour to convert that speed into 30 miles per hour. The letter that is commonly used for speed is v, for velocity. People almost always use d for distance and t for time. The formula for determining speed is:

a. $v = t + d$

b. $v = d \cdot t$

c. $v = d / t$

d. $v = d - t$

e. $v = t / d$

This formula gives us the average speed of an object over time t.

20. Now that you know the formula for determining speed, rearrange it to get a formula for the distance traveled:

a. $d = v + t$

b. $d = t / v$

c. $d = v \cdot t$

d. $d = v / t$

e. $d = v - t$

21. The following problem may seem trivial but it is intended to help you acquire the skill to handle a matter of life and death in Chapter 8, so pay close attention. Your friend Alyssa is coming over to your house. She lives exactly 1 mile away, and she is walking at a steady speed of 2.4 miles per hour. What is Alyssa's speed in miles per **minute**?

22. You're eager to see Alyssa, and after 20 minutes you wonder if she'll be close enough that you can see her when you step outside. By the time you put on your shoes and get outside, another minute has passed.

How far has Alyssa walked in 21 minutes?

How far is she from **your** house after 21 minutes?

23. Alyssa is at distance d from your house after walking m minutes. This can be expressed by the formula

a. d = .04m + 1

b. d = 1 – .04m

c. d = .04 / m

d. d = .04m + 1

24. Suppose that instead of sitting at home waiting for Alyssa, you decide to leave your house at the same time she leaves hers. You walk toward her at a speed of 1.8 miles per hour. After walking m minutes at this speed, you will be at a distance d from **your** house, given by the formula:

a. d = .03m

b. d = .03 / m

c. d = 1 – .03m

d. d = .03m + 1

This exciting problem will be continued in another chapter when you meet up with Alyssa ....

The following questions are designed to teach you how to work with fractions in algebra.

Example: $3 \cdot \frac{1}{5} = \frac{3}{5}$

25. $3 \cdot \frac{1}{x} =$

26. $3 \cdot \frac{2}{7} =$

27. $3 \cdot \frac{2}{x} =$

28. $a \cdot \frac{1}{5} =$

29. $a \cdot \frac{1}{x} =$

30. $a \cdot \frac{b}{x} =$

31. $\frac{2}{5} \cdot \frac{1}{7} =$

32. $\frac{a}{b} \cdot \frac{c}{d} =$

33. $\frac{a}{b} \cdot \frac{a}{b} =$

34. $\frac{b}{2a} \cdot \frac{b}{2a} =$

35. This is the first quiz problem that requires an intermediate step, as we are going to accomplish dividing by a fraction by multiplying by its reciprocal. It is really tempting to just do that in your head and fill in the answer. That may work fine for arithmetic, but algebra can be confusing enough without doing things in your head. Grab a piece of paper and write the problem, the next step, and then the answer.

$\frac{2}{5} \div \frac{1}{2} =$

36. $\frac{a}{b} \div \frac{c}{d} =$

37. $\frac{2}{3} + \frac{1}{4} =$

38. When we have an unknown quantity, as in this problem, we may not be able to add the numerators of the fractions together and the + sign remains.

$\frac{2}{b} + \frac{1}{4} =$

39. $\frac{2}{b} + \frac{1}{d} =$

40. $\frac{a}{b} + \frac{1}{d} =$

41. $\frac{a}{b} + \frac{c}{d} =$

42. $\frac{b}{4a} + \frac{c}{a} =$

43. Rearrange: $\frac{x}{y} = z \qquad y =$

44. $\frac{x}{y} = z \qquad x =$

45. $\frac{a}{9p} = x \qquad a =$

Help for Quiz

Quiz 5, problems 1 – 8

These problems check that you understood the basic principle explained in the chapter. If they seem difficult, replace the unknowns with real numbers and check carefully what happens to these numbers as you multiply and divide. You should see that some of the quantities can be removed without affecting the final answer.

Quiz 5, problem 9

Recall that the perimeter of a figure is the distance all the way around. Draw the two rectangles described, and carefully add up all four sides for each one. Can you rearrange these terms and add some of them together to make a simpler expression? After you add the terms you should be able to see how much bigger the second perimeter is. If it's not obvious just use some real numbers for the two unknowns.

Quiz 5, problems 10 – 16

Here both terms should be divided by the largest possible amount. That may be a number, an unknown, or both. Remember that a^2 is just a times a, so it can be divided by a. Multiply your expression back out to make sure it is still the same. If you have checked that but your answer is wrong, you most likely have overlooked something more that you could divide by. If you are stuck try replacing the unknowns with real numbers and then look for the largest possible number you can divide by.

Quiz 5, problem 17

This problem is related to the portfolio assignment for Chapter 4. Make sure you understand how to use the distributive property when there are three terms involved. You should then be able to reverse the process. Notice that while your first impulse may be to divide by 4, the last term cannot be divided by 4 evenly. Find something that you can divide every term by.

Quiz 5, problem 19

If a car travels 120 miles in 2 hours, we say that its speed is 60 miles per hour. We get the number 60 by dividing 120 by 2. If that same car covered a distance of 120 miles in 3 hours, its speed would be only 40 miles per hour (120 ÷ 4). In general, to find the speed we divide the .... by the ....

Quiz 5, problem 20

If you know that $\frac{6}{3}$ = 2, you should be able to rearrange that to say that 6 = ...

Quiz 5, problem 21

In this chapter you learned how to change something to a different unit using a conversion factor. You need to convert miles per hour to miles per minute. To replace "hours" by "minutes" you need to multiply by the right conversion factor. Try either 60 minutes/hour or 1 hour/60 minutes. One of these will magically remove "hours" and leave "minutes" in its place.

Quiz 5, problem 22

To figure out how far from your house Alyssa is after 21 minutes, notice that "your" is emphasized in the question. First you need to figure out how far Alyssa walks in 21 minutes. Once you know her speed that should not be too difficult. Next consider where she is in relation to your house after she has walked that distance. When you understand how to do that you can find a general formula that describes how far Alyssa is from your house after m minutes

Quiz 5, problem 24

This formula is easier to find because you are walking away from your house rather than toward it like Alyssa. You can just use a general formula for distance traveled.

Quiz 5, problems 25 – 36: Working with Fractions in Algebra

http://www.wtamu.edu/academic/anns/mps/math/mathlab/beg_algebra/beg_alg_tut3_fractions.htm

Quiz 5, problem 38

Algebra is a generalization of what you do with numbers. Look carefully at what you did with problem 37. How did you get your answer? I would expect that you were thinking: 2 times 4 is 8, and 3 times 1 is 3, so for the top part I get 8 + 3 which is 11. Once you start using unknowns the answer won't look quite as simple. The unknown just stays there, creating a more complex expression. The part on the bottom is still 12, but the part on the top should be equivalent to 8 + 3 (the + sign stays because you can't do anything more to add the terms). Once you have found this expression use a real number for the unknown a to check your work.

Quiz 5, problem 39

To add fractions, you were taught to find the lowest number that both bottom numbers go into. Sometimes there is no number for this other than the product of the two denominators. When 3 and 4 are on the bottom, you must change both fractions to have 12 as the denominator. For b and d, the only way you can get the same denominator is if you multiply them just like you did with 3 and 4. Remember to keep the letters in alphabetical order.

Quiz 5, problem 42

You may still be a little shaky on how to add fractions with real numbers. Remember that we need a new number on the bottom that both 4a and a go into. Let's go back to some real numbers. This time we'll pick 2 as the sample number for a (recall that 1 and 2 do not make good sample numbers). 4a is now 12, so the two numbers on the bottom are 12 and 3. what would you normally do with that? say you were doing 5/12 + 10/3? Once you know how to get the right denominator, go back to the question with the unknown and do it the same way. If you want to do it the lazy way instead, by multiplying the two bottom numbers, that works too. We get $4a^2$ on the bottom, and ab + 4ac on the top. Then be divide both top and bottom by a to get the fraction in simplest terms.

Quiz 5, problems 43 – 45

These questions allow you to practice rearranging something like 6/3 = 2 quickly like you learned in this chapter. Look back at the text if you are having difficulties. Remember that those unknowns stand for real numbers, so just replace them with some sample numbers if you get confused.

Chapter 6: Balancing Equations

Have you ever seen a picture of Lady Justice, the blindfolded lady who holds up the old–fashioned scales? For those of you who haven't, make sure to look her up online.

Okay, create a picture of those scales in your mind. Get rid of the lady, because we'll be using some very heavy stuff and she knows a lot of lawyers. Let's start with something simple. Put 4 + 3 on one side, and 7 on the other. When I was in kindergarten, they actually had real plastic numbers like that that were just the right weight to balance on a little set of scales. Now the scales are balanced.

$4 + 3 = 7$

Let's add something to both sides, like 5:

$4 + 3 + 5 = 7 + 5$

The scales still balance, because we have added the same amount to both sides.

We can also subtract something, and provided we subtract the same thing on both sides the scales continue to balance:

$4 + 3 = 7$

$4 + 3 - 4 = 7 - 4$

$4 - 4 + 3 = 7 - 4$

$3 = 7 - 4$

Now I'm going to make both sides twice as heavy by multiplying each side by 2:

$4 + 3 = 7$

$2(4 + 3) = 2 \cdot 7$

$2 \cdot 4 + 2 \cdot 3 = 2 \cdot 7$

$8 + 6 = 14$

As expected, we can multiply both sides of an equation by the same number without changing the truth of the equation.

If multiplication works, so should division. Let's divide both sides by 7:

$4 + 3 = 7$

$$\frac{4+3}{7} = \frac{7}{7}$$

Notice that both numbers on the left need to be divided by 7. That can be very easy to forget when there are a lot of things on one side of an equation.

$$\frac{4}{7} + \frac{3}{7} = 1$$

The important part to remember is that you may add the same thing to both sides of an equation, subtract the same thing from both sides, multiply both sides by the same thing, or divide them by the same thing.

My little kindergarten scales didn't come with unknowns, but let's imagine them now. We clear off the scales and take out the letter x. Now we try different numbers on the other side until the scales are balanced. As it turns out, the scales balance when the letter x is on one side, and the number 5 is on the other side. We conclude that $x = 5$ for this particular x. Not too hard so far. Next, add the number 2 to the side with the x on it. The scales tip as this side gets heavier. Add a 2 to the other side also, to balance the scales:

$x = 5$

$x + 2 = 5 + 2$

Now replace the numbers 5 and 2 on the right with the number 7:

$x + 2 = 7$

The scales balance again. Our scales tell us that $2 + x = 7$. In this equation x is still 5 but there is some extra stuff on both sides.

Let's test your imagination a little. Enlarge your scales and add an elephant to the side with the 2 and the x. Then add an identical elephant to the other side. Your scales should still balance [if they don't, go take a long break and don't think about elephants for at least a week]. We conclude that

an elephant + 2 + x = 7 + an elephant

If you just joined us after the elephants were added, you could easily figure out the value of x. First, you carefully remove the elephants from both sides, followed by the number 2:

$2 + x = 7$

$2 + x - 2 = 7 - 2$

$2 - 2 + x = 5$

$x = 5$

To remove the number 2, we subtracted 2 from both sides. That is really the same as adding −2 on both sides. Because adding −2 to 2 makes the 2 disappear, −2 is called the **additive inverse** of 2.

Usually people do not add elephants to equations; they add unknowns like x, y, a, b etc. Your job will be to carefully remove this extra stuff to determine the value of a particular unknown. Algebra students actually spend a lot of time doing this with equations. It is not hard, but maybe because it is so easy it is also easy to make sloppy mistakes. I do it all the time. This is why it is useful to use a pencil for algebra rather than a pen.

Let's see how the author of a math textbook might create a simple equation for you to solve.

$x = 10$

$5 + x = 15$

$x + 5 + x = 15 + x$

$x + x + 5 = 15 + x$

$2x + 5 = 15 + x$

To solve this last equation means to find the value of x that would make the equation true. The only value for x that will do that is 10. All other values will cause the equation to be false. You can find the value of x by carefully removing things from both sides of the equation. At each step you must remove exactly the same thing from both sides. Let's start with the unknown x:

$2x + 5 = 15 + x$

$2x + 5 - x = 15 + x - x$

$2x - x + 5 = 15$

Taking x away from 2x leaves x:

$x + 5 = 15$

To do this on paper, you would write –x underneath the equation on both sides:

$$\begin{array}{l} 2x + 5 = 15 + x \\ \underline{-x \qquad\qquad\quad - x} \\ \;x \;\; + 5 = 15 \end{array}$$

Now we can just remove 5 from both sides to get $x = 10$.

Once we have the answer, we substitute it back into the original equation to see if we were right:

$2x + 5 = 15 + x$

$2(10) + 5 = 15 + 10$

Using parentheses here allows us to put the 10 directly where the x was in the equation. When there is a number in front of parentheses you are supposed to multiply.

$2 \cdot 10 + 5 = 15 + 10$

$20 + 5 = 25$

And yes it works.

Hmm, this doesn't look so hard, so why don't we take a shortcut and remove both the x and the 5 at the same time? Yes it has been tried, but because of the way the human brain works this approach causes errors a lot of the time. **Don't do it!** Only change one thing at a time, because it is much easier to do things right in the first place than to try to figure out where you went wrong.

We can do even more with equations to help us find out what an unknown quantity is. None of these things are hard either. For example, suppose I'm playing with my toy scales. You walk in and see that I have 4 copies of the letter x on one side of the scales, and the number 20 on the other. Even if you didn't know that x is 5 in this toy set of letters and numbers, it wouldn't take much to figure it out. You just divide both sides by 4.

$4x = 20$

$$\frac{4x}{4} = \frac{20}{4}$$

$x = 5$

Dividing by 4 is the same as multiplying by $\frac{1}{4}$. Because multiplying by $\frac{1}{4}$ makes 4 disappear, $\frac{1}{4}$ is called the **multiplicative inverse** of 4. A number times its multiplicative inverse is 1, and that 1 doesn't really show because something multiplied by 1 is just itself. Now that you need to get rid of things in equations, you will want to pay close attention to additive and multiplicative inverses.

For any equation containing an unknown you can just divide both sides by the same number [or multiply by the appropriate multiplicative inverse]. If 3b = 9, then divide by 3 or multiply by $\frac{1}{3}$:

$\frac{3b}{3} = \frac{9}{3}$, so b = 3. Division is useful ***after*** you have done as much as you can at subtracting all the extra stuff in an equation. For example:

$8x + 5 = 6x + 13$

$8x + 5 - 6x = 6x + 13 - 6x$

$2x + 5 = 13$

$2x + 5 - 5 = 13 - 5$

$2x = 8$

Now I would have to subtract x to get a single x on the left side, but the right side doesn't have an x to take away! Removing x on both sides will not help you get an equation that says x = the answer: $2x - x = 8 - x$, so you end up with $x = 8 - x$. That is not wrong, but it isn't helpful.

Instead, divide both sides by 2:

$$\frac{2x}{2} = \frac{8}{2}$$

$x = 4$

At this point it would be a good idea to stop and see how this all works. Several websites supply algebra balance scales. Here are some:

[Algebra Balance Equations (hoodamath.com)]
[Algebra and Balance (animated-mathematics.net)]

Note that to remove a negative quantity, we need to add a positive one that cancels it out.

Just one word of caution here: inexperienced students sometimes try to turn 4x into x by subtracting 3 from 4x, or even subtracting the 4 "to get rid of it". If you are used to using balance scales you will see that this doesn't work. There are only two ways to turn 4x into x: either subtract 3x, or divide by 4.

$4x - 3x = x$

$\frac{4x}{4} = x$

It seems like whenever I unfold a map it ends up upside down. It also seems like whenever I try to carefully solve an equation I end up with minus signs on both sides. For example, take

$5x + 2 = 6x - 4$

First I subtract 6x from both sides:

$5x + 2 - 6x = 6x - 4 - 6x.$

$-x + 2 = -4$

Then I remove 2 from both sides to get $-x = -6$

If this happens to you, don't worry. Just divide both sides by −1. A negative number divided by a negative number becomes positive, so this trick simply takes away the − signs to give you $x = 6$. You could also multiply both sides by −1 to get the same result.

Going back to $5x + 2 = 6x - 4$, I see that it would have been smarter to subtract 5x from both sides:

$5x + 2 - 5x = 6x - 4 - 5x$

$2 = x - 4$

$2 + 4 = x - 4 + 4$

$6 = x$

Yes it is fine if the x ends up on the right side of the equation. You can just turn it around if you feel picky.

x = 6

This "turning around" has a name, as do most of the operations we perform to balance equations. It is called the **symmetric property of equality**. You can read about more of these names below:

Properties of Equality

These properties apply to numbers and numerical quantities, like length or weight. You can use them to justify algebraic operations like subtracting something from both sides of an equation, or dividing both sides of an equation by the same number.

Addition Property of Equality	If a = b, then a + c = b + c
Subtraction Property of Equality	If a = b, then a – c = b – c
Multiplication Property of Equality	If a = b, then a · c = b · c
Division Property of Equality	If a = b, then a ÷ c = b ÷ c (c ≠ 0)
Reflexive Property of Equality	a = a
Symmetric Property of Equality	If a = b, then b = a
Transitive Property of Equality	If a = b and b = c, then a = c
Substitution Property of Equality	If a = b, then b can replace a in any expression

The **Distributive Property** says that a(b + c) = ab + ac. It is also used to justify adding like terms, like 2x + 3x = 5x. That works because you can write 2x + 3x as (2 + 3)x = 5x.

Multiple Solutions or No Solutions

Sometimes you may get a surprise when you try to find the value of x from an equation. Look at this example:

5x = 5(x + 3) – 15

First we use the distributive property, and then we simplify things.

5x = 5x + 15 – 15

5x = 5x

Hmm, that looks suspicious. Of course 5x is equal to 5x. Subtracting 5x from both sides produces something equally obvious:

$0 = 0$

This is not a particular solution; it is just a true statement. What that means is that the original equation is true for any value of x. We can pick any number for x and get a true statement – try it out!

At other times, there is just no way to find any value for x that fits.

$3x + 4 + x = 6x - 9 - 2x$

$3x + x + 4 = 6x - 2x - 9$

$4x + 4 = 4x - 9$

That doesn't look promising. Subtract 4x from both sides:

$4 = -9$

This is a false statement. Picking different values for x here will not help that. No value for x will make the equation true. There are no solutions.

Equations Containing Fractions

Eek, there is a fraction in my equation! That's like having a fly in your soup. If this is how you feel, you may want to get rid of the fraction first, before doing anything else. Look at this equation:

$\frac{1}{2}a = 3$

Notice that $\frac{1}{2}a$ really means $\frac{1}{2} \cdot a$, which is $\frac{1}{2} \cdot \frac{a}{1} = \frac{a}{2}$

If $\frac{a}{2}$ bothers you, you can get rid of "divided by 2" by multiplying both sides by 2. The number 2 is the multiplicative inverse of $\frac{1}{2}$, so multiplying by 2 will remove $\frac{1}{2}$ from your equation.

$2 \cdot \frac{1}{2} a = 3 \cdot 2$

$a = 6$

If the equation has more terms, like $3x - \frac{1}{4}x = 15 - x$, make sure you multiply *everything* by the part that you want to get rid of, "divided by 4" in this case. Using parentheses really helps with that.

$3x - \frac{1}{4}x = 15 - x$

$4(3x - \frac{1}{4}x) = 4(15 - x)$

$4 \cdot 3x + 4 \cdot \frac{1}{4}x = 4 \cdot 15 - 4 \cdot x$

$12x - x = 60 - 4x$

Now that the icky fraction is gone it is easier to solve for x. Start by simplifying:

$11x = 60 - 4x$

Then add 4x to both sides:

$11x + 4x = 60 - 4x + 4x$

$15x = 60$

That is as far as things go for adding or subtracting. Divide both sides by 15 to get the answer:

$\frac{15x}{15} = \frac{60}{15}$

$x = 4$

On the other hand, if you are not afraid of fractions you can leave them until the end:

$3x - \frac{1}{4}x = 15 - x$

$2\frac{3}{4}x = 15 - x$

$2\frac{3}{4}x + x = 15 - x + x$

$3\frac{3}{4}x = 15$ Convert the mixed number to a fraction:

$\frac{15}{4}x = 15$

$4 \cdot \frac{15}{4}x = 15 \cdot 4$

$15x = 60$

$\frac{15x}{15} = \frac{60}{15}$

$x = 4$

There are a lot of steps here, and you may find it easier to take a shortcut. Let's back up to $\frac{15}{4}x = 15$. That really means $\frac{15}{4} \cdot x = 15$. We can get rid of $\frac{15}{4}$ in one step if we divide by $\frac{15}{4}$ on both sides. If you remember your fractions, you know that dividing by a fraction is the same as multiplying by its reciprocal (flip it around). So, we can multiply both sides of the equation by $\frac{4}{15}$. $\frac{4}{15}$ is the multiplicative inverse of $\frac{15}{4}$.

$\frac{15}{4}x = 15$

$\frac{4}{15} \cdot \frac{15}{4}x = 15 \cdot \frac{4}{15}$

$x = 4$

That was faster, but you can do it either way 😊.

Sometimes the fractions in an equation don't have a common denominator:

$\frac{2x}{3} - 5 = \frac{x}{6} + 2\frac{1}{2}$

Add 5 to both sides:

$\frac{2x}{3} = \frac{x}{6} + 7\frac{1}{2}$

Subtract $\frac{x}{6}$ from both sides:

$\frac{2x}{3} - \frac{x}{6} = 7\frac{1}{2}$

$\frac{2x}{3} - \frac{x}{6} = \frac{15}{2}$

Create a common denominator: $\frac{2x}{3} = \frac{4x}{6}$

$\frac{4x}{6} - \frac{x}{6} = \frac{15}{2}$

Subtract just like you would for a regular fraction:

$\frac{4x-x}{6} = \frac{15}{2}$

$\frac{3x}{6} = \frac{15}{2}$

$$\frac{x}{2} = \frac{15}{2}$$

Multiply both sides by 2 to find the answer:

$$x = 15$$

Alternatively, you can eliminate all of the fractions by multiplying by 6 on both sides. Let's go back to the start:

$$\frac{2x}{3} - 5 = \frac{x}{6} + 2\frac{1}{2}$$

$$\frac{2x}{3} - 5 = \frac{x}{6} + \frac{5}{2}$$

$$6 \cdot \left(\frac{2x}{3} - 5\right) = \left(\frac{x}{6} + \frac{5}{2}\right) \cdot 6$$

$$\frac{12x}{3} - 30 = \frac{6x}{6} + \frac{30}{2}$$

$$4x - 30 = x + 15$$

$$4x = x + 45$$

$$3x = 45$$

$$x = 15$$

Cross-Multiplying

If there are fractions on both sides of the equation, we could multiply twice to eliminate them. For example, let's solve

$$\frac{1}{5x} = \frac{3}{4}$$

First we multiply both sides by 5x, to get

$$5x \cdot \frac{1}{5x} = 5x \cdot \frac{3}{4}$$

$$1 = \frac{5x}{1} \cdot \frac{3}{4}$$

$$1 = \frac{5x \cdot 3}{1 \cdot 4}$$

$$1 = \frac{15x}{4}$$

Then we eliminate the second fraction by multiplying both sides by 4. We get

$$4 \cdot 1 = 4 \cdot \frac{15x}{4}$$

$$4 = 15x$$

Now divide both sides by 15 to get the answer:

$$\frac{4}{15} = \frac{15x}{15}$$

$$\frac{4}{15} = x$$

Phew, that was a lot of work! Fortunately there is a shortcut. When there are two fractions separated by an equals sign you can just cross–multiply. Multiply the top part of one fraction by the bottom part of the other fraction, and the other way around. That has the same result as multiplying twice, but it is much faster. This method takes advantage of the following fact:

If $\frac{1}{4} = \frac{2}{8}$, then 1 times 8 equals 2 times 4. Try it out with some equivalent fractions to see that it works.

You may have seen cross–multiplication before, but now that you know algebra you can understand why it works.

Suppose that we have some generic fraction $\frac{a}{b}$ that happens to be equal to some other fraction $\frac{c}{d}$:

$\frac{a}{b} = \frac{c}{d}$ To get rid of "divided by b", we multiply both sides by b:

$b \cdot \frac{a}{b} = b \cdot \frac{c}{d}$

$a = \frac{b}{1} \cdot \frac{c}{d}$

$a = \frac{bc}{d}$ Now multiply both sides by d:

$d \cdot a = d \cdot \frac{c}{d}$

$ad = bc$

So for any two fractions where $\frac{a}{b} = \frac{c}{d}$, we can just cross–multiply and say that ad = bc.

To solve $\frac{1}{5x} = \frac{3}{4}$, cross–multiply to get:

$5x \cdot 3 = 1 \cdot 4$

$15x = 4$

Notice that cross–multiplying eliminates both fractions and gives you a new, equivalent equation that is much easier to solve. To finish up, divide both sides by 15:

$x = \frac{4}{15}$

That is a lot faster than what we did before.

To create your own example, take a fraction like $\frac{a}{b}$ and multiply both the top and the bottom by the same number to produce an equivalent fraction. Let's use the number 3:

$\frac{a}{b} = \frac{3a}{3b}$. Now cross–multiply to get a · 3b = b· 3a, which says that 3ab = 3ab.

Using the variable x instead of the number 3, we can see the general result:

$\frac{a}{b} = \frac{ax}{bx}$ so a · bx = b · ax. Put that in alphabetical order to get abx = abx.

Percentages

Once you know how to cross–multiply, you can use it to solve percentage problems. Suppose you need to find a number that is 5% of 60. You may remember from arithmetic that you can multiply: $\frac{5}{100} \cdot 60 = 0.05 \cdot 60 = 3$

Algebra makes these percentage problems easier because you can just use x for the unknown number, and set things up as a proportion:

$$\frac{5}{100} = \frac{x}{60}$$

Cross–multiply first:

$$5 \cdot 60 = 100 \cdot x$$

$$300 = 100x$$

Then divide by 100 to get x:

$$\frac{300}{100} = \frac{100x}{100}$$

$$3 = x$$

This same technique works for problems that are written the other way around, such as: 3 is what percentage of 60?

Now the percentage is unknown, so call it x:

$$\frac{x}{100} = \frac{3}{60}$$

$60x = 300$

$x = 5$

The answer is 5%.

If the problem asks: 3 is 5% of what number, then we call that number x. This number x corresponds to 100%, while the number 3 corresponds to 5%:

$$\frac{5}{100} = \frac{3}{x}$$

$5x = 300$

$$\frac{5x}{5} = \frac{300}{5}$$

$x = 60$

To Divide or Not to Divide

As we have seen, you can divide something by a positive number, a negative number, and even an unknown number. The one thing that you cannot do, however, is to divide by zero. This actually makes sense when you think about it. In the chapter on negative numbers we were able to divide something among –3 people by really stretching our imagination. It is much easier to find 0 people, but look what happens when you try to divide something among them – you can't even get started because there is no one there. Another problem would be that things have to multiply back out:

$\frac{6}{2}$, and $3 \cdot 2 = 6$.

$\frac{6}{0} = ??$ $\quad 0 \cdot ?? = 6$. There is nothing that you can multiply 0 by to get back to 6.

When an expression potentially calls for dividing by zero, such as $\frac{x}{3-x}$, we say that the expression is **undefined** when x = 3.

Normally this would be a simple rule, but when you are dealing with unknowns zero can come up unexpectedly. You'll divide by x [or some other unknown] and suddenly, before you know it, you've divided by 0! Don't do this, because bad things will happen....

Here is one of those:

$4x = 4x^2$

First I divide by 4 on both sides to get $x = x^2$. That looks a bit odd, but not to worry, I can just divide by x on both sides. On the left side, x divided by x is just 1, and on the right side x^2 divided by x is x. I write 1 = x, and now I've solved the problem, right? Checking my solution, I find that x = 1 fits the original problem just fine, so that 4 = 4. Unfortunately, I've just lost at least half of whatever points this question was worth, or perhaps all if my teacher is disgusted with my failure to use proper algebra methods. There are two answers to this problem: x = 0 or x = 1.

The smart thing to do is to **consider that x could be 0 before you divide by it**:

$x = x^2$

x = 0 (yes that fits) **or**

$\frac{x}{x} = \frac{x^2}{x}$

1 = x

You can also move x to the other side of the equation for a more elegant solution:

$0 = x^2 - x$

Now we can factor, which is similar to a division by x except that the x sticks around:

$0 = x(x - 1)$

Here we have two things that multiply to give 0: x and x − 1. If you think about this a bit, you'll realize that the only way that you can multiply two different numbers together and get 0 is if at least one of those numbers is 0. That means that either x is 0 or x − 1 is 0. If

$x - 1$ is zero then x must equal 1.

Tougher Balancing Acts

Sometimes equations have more than one unknown. We already saw such equations in the previous chapter, like $\frac{a}{b} = c$, and learned to rearrange them into $a = bc$ or $\frac{a}{c} = b$. Now you can use basic algebra principles to accomplish this same rearrangement.
If $\frac{a}{b} = c$, then we can multiply both sides by b to get rid of "divided by b", like this:

$$\frac{a}{b} = c$$

$$b \cdot \frac{a}{b} = b \cdot c$$

$$a = bc$$

Now we could divide both sides by c:

$$\frac{a}{c} = b$$

If there is more than one unknown, it is usually impossible to get a nice answer like a = 5, and nobody will expect you to do that. You will just be asked to solve the equation for one of the unknowns while leaving the other ones. Perhaps the value for the remaining unknowns will be available at some later time, and then you'll have a specific answer. It may also happen that one or more of the other unknowns cancel out. Most of the time though, your answer will just show the relationship between various unknowns. Let's look at an example:

$ax - y = 7$, solve for x.

We still use the same strategies. Try to get the unknown that you want all by itself on one side of the equation. Here we can do that by adding y to both sides to get

$$ax = 7 + y$$

To get rid of the a in front of the x, we divide both sides by a:

$x = \frac{7 + y}{a}$

That's as far as we can get without knowing the values of a and y. These kinds of manipulations are useful to rearrange formulas, like the formula for the circumference of a circle, C = 2πr. If we know C and want to find r, we can rearrange the formula by dividing both sides by 2π:

$\frac{C}{2\pi} = r$

There is an important trick that you need to know when the unknown you want is present in more than one term. You have already seen this trick, but you might not think of using it in this situation.

$ax - 3 = b - 5x + 4$, solve for x.

Don't panic! Even if the equation looks hard, you can still rearrange it any way you want using your existing algebra skills. Let's work on this until all the terms with the unknown we want are on one side, and the other stuff is on the other side. First add 3 to both sides:

$ax = b - 5x + 7$

Now we could divide both sides by a, but that will not help us to solve for x. Instead, add 5x to both sides so that we can get all the terms with x on the left:

$ax + 5x = b + 7$

Next we need to isolate x. Here factoring comes in handy:

$x(a + 5) = b + 7$

Don't let the parentheses scare you. The left side of the equation just says "x times the whole thing inside the parentheses". That means that we can simply divide by the whole thing inside the parentheses:

$x = \frac{b + 7}{(a + 5)}$

The parentheses are now redundant, so we can just remove them

$x = \frac{b + 7}{a + 5}$, and that's the final answer.

Troubleshooting

I can pretty much guarantee you that I have made more errors in balancing equations than you have. Because I am very experienced at this I can offer you some advice on troubleshooting your equation solutions. [Note that such advice is not found in math textbooks, whose authors apparently never make mistakes.]

Anyway, here I will solve the following equation: $x - 3(x - 4) = 3x + 2$

Step 1: $x - 3x - 12 = 3x + 2$
Step 2: $-2x - 12 = 3x + 2$
Step 3: $-12 = 5x + 2$
Step 4: $-14 = 5x$
Step 5: $x = -14/5$

Now depending on the work you are doing, this answer could be worrisome because the answers to these simple equation problems are often nice whole numbers. Just to check if I'm right, I use my value for x in the original equation: $-14/5 - 3\ (-14/5 - 4) = 3(-14/5) + 2$. Simplifying I get:

$- 14/5 - 3(-14/5 - 20/5) = -42/5 + 10/5$
$-14/5 - 3(-34/5) = -32/5$
$-14/5 + 102/5 = -32/5$ and no that isn't going to work out.

I look over my original work again, really carefully this time. But no matter how hard I try, I just can't find my mistake. If this happens to you have two options. First, you could find someone else who knows algebra and get them to look over your work. Because it is not their work, they are likely to spot your mistake instantly. If you are unlucky and no such person is available, use your incorrect answer to tell you where the mistake is.

Step 5: $-14/5 = -14/5$

Step 4: $-14 = 5(-14/5)$

Step 3: $-12 = 5(-14/5) + 2$ or $-12 = -14 + 2$

Step 2: $-2(-14/5) - 12 = 3(-14/5) + 2$ or $28/5 - 60/5 = -42/5 + 10/5$ or $-32/5 = -32/5$

Step 1 (original equation): $-14/5 + 102/5 = -32/5$

All of these steps except the first one check out, which tells me that my error is somewhere between step 1 (where my answer doesn't work), and step 2 (where it does work). Knowing where the mistake is, I look again and see that I failed to correctly multiply $-3(x - 4)$, which should be $-3x + 12$ instead of $-3x - 12$.

Does this seem like a lot of work? Of course it does. Good algebra students aren't just "good at algebra"; they put in a lot of hard work. And after you've worked this hard to find your mistake, you'll learn not to make it again . Other common errors are adding something on one side but subtracting it on the other, and failing to multiply each and every term by the same amount when you are trying to get rid of a fraction. I'll leave you to make those mistakes on your own.

As we have seen, equations are very resilient things. You can pretty much do anything to them, so long as it's the same on both sides. It doesn't even have to be anything mathematical. If you set off a nuclear bomb on each side, nothing will be left on both sides. $0 = 0$, which is still a valid equation.

Movies from Khan Academy

Search for "khan Academy solving equations with one unknown".

Chapter 6 Practice

Tip: When you are solving equations, do only one thing at a time. For example, if you can quickly see that you should add 2 to both sides of the equation, and subtract x from both sides, don't be tempted to do it all at once. You are much more likely to make mistakes that way. Do one operation, write down the result, and then go on to the next one.

Use this website to practice: http://www.shodor.org/interactivate/activities/AlgebraQuiz/ You can select "no time limit". Start on "easy" and keep going until you can do a few of the "hard" problems. Try all the boxes **except for "quadratic"** – we'll get to that one later.

Portfolio Chapter 6

Now would be a good time to create some of your own equations. Make as many as you like, and try to solve them. Make sure to check that your answers work, and that you haven't overlooked the possibility that x could be zero.

Were you able to solve all of your equations? Why or why not?

Don't forget to add to your portfolio page if you learned something while doing the quiz.

Chapter 6 Quiz

Algebra's favorite unknown, x, gets itself into some difficult situations in the course of its mathematical duties. Extract x from the following equations:

1. $x + 4 = 8$

$x =$

2. $x - 12 = 23$

$x =$

3. $-2x = 10$

$x =$

4. $14x - 4 = 24$

$x =$

5. $4x - 6 = 2x + 4$

$x =$

6. $8x - 2 = 3x + 8$

$x =$

7. $3x + 10 - 2x = -5x - 2 + 3x - 3$

$x =$

8. $6\,(x + 6) = -6x$

$x =$

9. $3 - \frac{1}{3}x = 12$

$x =$

10. $-\frac{3x}{4} + 2 = 5 - x$ $x =$

11. $\frac{x^2-4x}{x} = 0$ [Here we have to specify that x cannot be 0, so that we will not divide by 0]

$x =$

12. $\frac{4}{5} = \frac{12}{x}$

$x =$

13. $ax + 6 = 6x + a$

$x =$ [This problem has been carefully constructed so that it is possible to find a numerical value when you solve for x. For this particular value of x, a can be anything.]

14. In arithmetic you learned how to find a percentage of a number:

5% of 60 is 0.05 x 60, which equals 3.

Now try to solve a question that works the other way around: 12% of what number is 108? In this case the number is unknown, but now that you know algebra you could just call it n and use the same method you learned before.

15. A number plus 5 times that number is 90. What is the number?

16. Find the value(s) of y:

$y = 2y^2$

Help for Quiz

Quiz 6, problem 10

Remember that when you multiply by 4 to get rid of the fraction, you must multiply everything by 4 otherwise the two sides of the equation will no longer be equal.

Quiz 6, problem 11

This problem looks scarier because it has an exponent, but that really doesn't make it more difficult. Just remember that the entire thing above the division line must be divided by what is underneath. That means you have to divide x^2 by x, and also divide 4x by x. Since x^2 is just x times x, it should be possible to divide by x. If you are not sure how, just try some real numbers. (3 times 3) divided by 3, or (4 times 4) divided by 4. Once you have that figured out, divide 4x by x.

Quiz 6, problem 12

Cross–multiply this to get an equation without fractions that you should be able to solve.

Quiz 6, problem 13

Look at the example in the text again if you are having difficulties. The strategy to use is to get all of the terms with x on the left, and all of the other things on the right. Use factoring (the distributive property in reverse) to isolate x, and then divide. Now look at the two things you are dividing. Even though it looks complicated, what must be the result of this division? Use a sample number for a to see that the answer does not contain any unknowns; it is just a number.

Chapter 7: Inequalities

Most people "get" the idea of equations. This thing is equal to that thing, or all of this stuff = all of that stuff, is something that we encounter in our daily lives. Inequalities happen in our lives too. "Jimmy has more candies than I do," is connected to a sense of unfairness, and scientists have discovered that our brains are very sensitive to even small differences when someone else has more of something than we do. $a > b$ rarely gives us much trouble in terms of understanding. This tells us that a is bigger than b, which makes sense in a general way. However something like $x > 5$ may not come as naturally. x is bigger than 5? Well how much bigger is it? What is x? Humans count, weigh, measure and calculate to get rid of such uncertainties. Algebra carelessly permits them to exist. If $x > 5$, x could be 5.001, or 6, or 1000,000, or even more. On the other hand, x could **not** be 5, 0, or –1000,000. We have to be satisfied with knowing just this one fact about x, which is that it is bigger than 5.

An expression like $z + g > g$ that we saw in Chapter 1 is an inequality. For the most part, inequalities can be managed just like equations. After all, if we know that $z + g > g$ then $5 + z + g > g + 5$. This is equivalent to giving Zorg and Grom 5 more shells each. Adding 5 shells to both sides does nothing to change the truth of the inequality. In the same way, you can subtract something from both sides of an inequality. In the case of $z + g > g$, an interesting thing to subtract is g, as it is already present on both sides. $z + g - g > g - g$, therefore $z > 0$. Think about this and make sure you believe that it is always true.

Now, let's give both Zorg and Grom twice as many shells. It is still true that $2z + 2g > 2g$. So far it works just like it does with equations. In fact, we don't experience any problems manipulating inequalities at all, until we try to multiply or divide by a negative number. Look at the following simple inequality: $3 < 5$. Dividing both sides by -1, we get $-3 < -5$. This is a problem, because -5 is considered a smaller number than -3, since it is more negative and lies to the left of -3 on the number line. **So, if you ever find yourself having to divide or multiply an inequality by a negative number, remember to turn your symbol around: $3 < 5$ becomes $-3 > -5$.**

If you have a problem like $3x + 3 < x + 5$ just follow the same steps you would for solving an equation.

$3x + 3 < x + 5$

$2x + 3 < 5$

$2x < 2$

$x < 1$

How will you know if this answer is correct? The best approach is to take the first whole number that is actually smaller than 1, which is 0 in this case, and check that the statement would actually be true. We get $3 < 5$, which is true. Now take the closest incorrect answer, $x = 1$. If we are right that x should be less than 1, substituting 1 into the inequality should give an incorrect statement. Now we get $3 + 3 < 1 + 5$, which is clearly not true, so our answer should be right. Notice that $3 + 3$ is equal to $1 + 5$, because if the problem had been an equation rather than an inequality the solution would have been $x = 1$.

Just like with equations, x sometimes ends up on the wrong side.

$2x - 3(x - 1) < 5x + 2$

Usually it is best to apply the distributive property first, because multiplication is higher up in the order of operations. Be very careful because you are multiplying by −3, and −3 times −1 is positive 3:

$2x - 3x + 3 < 5x + 2$

$-x + 3 < 5x + 2$

$3 < 6x + 2$

$1 < 6x$

$\frac{1}{6} < x$

This says that $\frac{1}{6}$ is smaller than x. x is on the large end of the inequality. Turn that around to say that x is larger than $\frac{1}{6}$:

$$x > \frac{1}{6}$$

To test your answer, try the closest correct whole number value, x = 1, as well as the closest incorrect whole number value, x = 0.

Dealing with inequalities may seem straightforward now as you read this, but problems involving inequalities can get rather complex. We will see such problems in a later chapter. For now, there is an excellent and thorough explanation of inequalities for you to read here: http://www.themathpage.com/alg/inequalities.htm In this text, the answers to questions are hidden by colored boxes, and after you find your own answer () you can move your mouse over the box to see if you were right. Caution: this text contains a lot of boring theorems. Do not memorize them; just try to understand why they work.

Practice for Inequalities

Practice solving inequalities here: http://www.math.com/school/subject2/lessons/S2U3L4GL.html#sm3. Don't go past the first set of problems that are there for you to do in "Step 4".

Portfolio Chapter 7

Just like you can make your own equations, you can create your own inequalities, and solve them of course.

How should you check your solutions?

Chapter 8: Word Problems

Word problems. We've all seen them. They have even been found on ancient papyrus scrolls, so they have probably been plaguing students since the dawn of mathematics. Word problems are intended to help us relate mathematics to real–life situations, but it is probably debatable whether they actually achieve their purpose. Some people find them easy to do while others get hopelessly tangled up in them. How well you do at solving a word problem does not just depend on your mathematical abilities. Many areas of your brain are involved in processing this type of problem. You have to read the whole problem carefully, create some sort of picture in your mind or on paper, and then identify and do the correct mathematical computations. However, just like with anything else your ability to solve word problems can be improved by knowing something about them, and by practicing.

What do we know about word problems? Well, most of them are short. They generally use only the minimum number of words required to set up a situation that you can picture in your mind. Descriptive adjectives are usually lacking. For example, a problem might start with: "A car is moving at a speed of 60 miles per hour," rather than "A red convertible sports car is speeding down a treacherous mountain road at 60 miles per hour." I'm not really sure if this is intended to minimize distractions, or just an indication that most mathematicians do not excel at creative writing. If you find that details help you create a mental picture you can of course just make them up. Another thing you may notice about word problems is that they are always politically correct. Persons in traditionally male–dominated occupations must be female in word problems. For example: "A construction worker is pouring a cement sidewalk. She ..." The most important thing about word problems however is that they tend to fall into specific categories. If you learn to recognize these categories you can usually deal with the problem more easily and efficiently.

Arithmetic Word Problems

"Arithmetic" word problems test the skills of addition, subtraction, multiplication, division, percentages and fractions. Simple geometry problems ask for area or perimeter, like the area of a window with a semicircle as its top part, or the length of a fence needed to enclose a certain area.

Brush up on your basic word problem skills by solving the **first 4 problems** of the THEA test: http://www.thea.nesinc.com/practice.htm. Select Mathematics at the bottom of the page. Do not do the remaining problems.

Equation Problems

An example of a "simple equation" word problem would be something like: "Harry is 5 years older than Corinne. If Corinne is 8 years old, how old is Harry? Let’s use h for Harry's age and c for Corinne's age. Unfortunately algebra doesn’t have a direct translation for “5 years older than”. Still, that doesn’t mean we can’t make our own translation because math exists for our convenience, not the other way around. You can see from the problem that Harry is older than Corrine, which means

$$h > c$$

Now, how much larger is h? Well, that would be 5 years. There is no official notation for that, but we can still indicate that the difference is 5 like this:

$$h \overset{5}{>} c$$

I made up this notation because it helps me see what is going on. This says that h is larger than c by 5 units. It is not an equation. To create an equation that will help us solve the problem, we can add 5 to the smaller side to make things equal:

$$h = c + 5$$ [Or you could subtract 5 from the left side to get $h - 5 = c$]

This equation can be solved because the problem gives us Corrine’s age, c, which is 8:

$$h = 8 + 5$$

$$h = 13$$

Harry is 13 years old. Make sure you put that value back into the original problem and check that it makes sense. If Harry is 13, he would indeed be 5 years older than Corinne.

You can use this strategy any time that a problem gives you two unequal quantities and tells you how much the difference between them is. Here is another example: Sally is 9 inches shorter than twice Roy's height. If Sally is 5 feet 3 inches tall, then how tall is Roy?

Again, the problem gives you two unequal amounts. The first one is Sally's height, which we will call s, and the second quantity is "twice Roy's height". If we call Roy's height r, then twice his height would be 2r. Compare s and 2r to see which one is greater.

$s < 2r$

It looks like s is smaller, because the problem says that Sally is "shorter than". What is the difference? Well, that would be 9 inches. s is smaller than 2r by 9:

$$s \overset{9}{<} 2r$$

Because s is the smaller amount, something has to be added to it to make both sides equal. Create an equation by adding 9 to the left side:

$s + 9 = 2r$

Because we know s we can put that value into the equation. Change 5 feet 3 inches into all inches to make sure there is one consistent unit. Five feet is $5 \cdot 12 = 60$ inches, plus 3:

$63 + 9 = 2r$

$72 = 2r$

Divide both sides by 2:

$r = 36$ inches, which is the same as 3 feet. Twice Roy's height would then be 6 feet, and at 5 feet 3 inches Sally is in fact 9 inches shorter than that.

More complex equation problems are those that translate into more than one equation. A common example is the following: "The sum of two numbers is 85. The larger number is 4 times the smaller number. What are the two numbers?"

Although you can solve this guessing and checking, that takes a long time and it is not what you are expected to do. Algebra was just made to solve things like this, so let's give it a try.

Using letters for the unknown numbers will allow us to see what is going on. Let's call the larger number ℓ, and the smaller number s. The first piece of information in the problem tells us that

$$\ell + s = 85$$

This is an equation that contains two unknowns, and there is no way of solving it. Fortunately we have some additional information. The larger number ℓ is 4 times the smaller number s. We can write that as

$$\ell = 4s$$

Because ℓ and 4s are equal, we can treat them as the same thing. Look again at the first equation, $\ell + s = 85$. Instead of ℓ, we can use 4s:

$$4s + s = 85$$

Now there is only one unknown, and it is easy to find.

$$5s = 85$$

$$s = \frac{85}{5}$$

$$s = 17$$

The smaller number is 17.

$\ell = 4s$

$\ell = 4 \cdot 17$

$\ell = 68$

The larger number is 68.

The trick of using 4s instead of ℓ is called **substitution**. This is one of the tricks that mathematicians use to solve a problem that contains or can be translated into two or more equations.

There are many variations of this problem. Here is another one: "The difference between two numbers is 16. The larger number is 6 more than 3 times the smaller number. What are the two numbers?"

To find the difference between two numbers, we subtract them:

$\ell - s = 16$

The second sentence compares two quantities: the larger number ℓ and "three times the smaller number, s." Three times s should be written as 3s. The number ℓ is larger than 3s, and the difference is 6:

$$\ell \overset{6}{>} 3s$$

Add 6 to the side with the smaller amount to get the equation:

$\ell = 3s + 6$

ℓ is 3s + 6, so use that in the first equation instead of ℓ:

$\ell - s = 16$

$(3s + 6) - s = 16$ Use parentheses when you make your substitution, in case the problem is more complicated, like maybe $3\ell + 5 = 20$. Without the parentheses you would fail to multiply everything that you substitute for ℓ by 3.

$2s + 6 = 16$

$2s + 6 - 6 = 16 - 6$

$2s = 10$

$s = 5$

Now find the larger number:

$\ell = 3s + 6$

$\ell = 3 \cdot 5 + 6$

$\ell = 21$

Proportion Problems

Another kind of problem that actually comes up surprisingly often in real life is the "proportion" word problem. This problem can be translated into two fractions with a single unknown. You can then cross–multiply to get the value of the unknown, or use another method to solve the equation [see "Balancing Equations – Equations Containing Fractions"]. An example would be:

A survey at a local high school showed that 4 out of 10 students like algebra. If the school has 450 students, how many like algebra?

We use the term "proportions" for these problems because they involve two sets of things that have the same relationship to each other in terms of size.

Translate this problem into two fractions, using x for the unknown number. 4 out of 10 can be written as the fraction $\frac{4}{10}$:

$$\frac{4}{10} = \frac{x}{450}$$

$$\frac{2}{5} = \frac{x}{450}$$

Cross–multiply:

$$2 \cdot 450 = 5x$$

$$900 = 5x$$

Divide both sides by 5:

$$x = 180$$

So, $\frac{4}{10} = \frac{180}{450}$. Since a fraction is really a division, it is easy to check your work. Just do 4 ÷ 10, which is 0.4, and then grab a calculator to check 180 ÷ 450. Both fractions should have the same value if they represent the same proportion.

You have already learned how to convert from one unit to another by using conversion factors. A slightly more cumbersome way to do the same thing is to set it up as a proportion. For example, suppose that you want to know how many feet there are in 15 yards. You already know that there are 3 feet in one yard, so create a proportion:

$$\frac{3\text{ feet}}{1\text{ yard}} = \frac{x\text{ feet}}{15\text{ yards}}$$

Note that it doesn’t matter whether you put feet on top of the fractions and yards on the bottom, or the other way around. Just don’t do it one way on the left and the other way on the right! Cross–multiply, leaving the units out of it to make your calculations easier:

$3 \cdot 15 = 1 \cdot x$

$x = 45$

There are 45 feet in 15 yards.

You can read more about proportions here:
http://www.math.com/school/subject1/lessons/S1U2L2GL.html.

A few chapters ago we encountered the "working together" word problem. As you may remember, the secret to solving these problems is to add the rates at which the people or machines are working. Once you have a total rate of say, 0.08 cars/minute, you can set up a proportion to find out how long it takes to wash one car when people are working at that rate:

$$\frac{.08\text{ cars}}{1\text{ minute}} = \frac{1\text{ car}}{x\text{ minutes}}$$

Ignore the units for a moment and cross-multiply to get

$.08x = 1$

$x = \frac{1}{.08}$

$x = 12.5$ minutes

Lucky for you, there is a problem like that in the quiz so you can try it out.

Percent Problems

As you may (or may not) remember from basic arithmetic, a percent is a one-hundredths part. 1% = $\frac{1}{100}$. A fraction is really a division, so you can divide 1 by 100 to get 0.01.

Any percentage can be converted to a decimal number: 4% = $\frac{4}{100}$ = 0.04 and 60% = $\frac{60}{100}$ = 0.6.

If you answer 18 out of 20 questions on a test correctly, you can determine your percentage grade by writing that as a fraction: $\frac{18}{20}$. You can use long division or grab a calculator to get the answer 0.9. 0.9 is $\frac{9}{10}$ or $\frac{90}{100}$, so that would be 90%. Because I picked some convenient numbers here, you can also use equivalent fractions: $\frac{18}{20} = \frac{18\text{x}5}{20\text{x}5} = \frac{90}{100}$.

Questions that ask you to find a percentage may be worded in different ways, such as "6 is what percent of 300?" To get the percentage, just write it as a fraction: $\frac{6}{300}$. This fraction is equivalent to $\frac{2}{100}$, or you can use a calculator to get 0.02. The answer is 2%. This answer makes sense because 1%, or one-hundredths part, of 300 would be 3. If 1% of 300 is 3, then 2% should be 6.

To find a fraction or percent of a number you need to use multiplication:

$\frac{1}{5}$ of 200 is $\frac{1}{5}$ times 200: $\frac{1}{5}$ x $\frac{200}{1} = \frac{200}{5}$ = 40. Notice that this multiplication by $\frac{1}{5}$ conveniently causes a division by 5 so we end up with a fifth part of 200. We can also use this multiplication trick with percentages because percentages are fractions. 1 divided by 5 is 0.2, or 20%. To find 20% of 200 we can multiply $\frac{20}{100}$ x 200, or 0.2 x 200. The answer is still 40.

Algebra extends basic arithmetic by asking questions like "15% of what number equals 90?" Now we no longer have a simple division or multiplication involving two numbers. Instead, the number that you are supposed to take 15% of is unknown. Unknowns are the specialty of algebra, and you can just use a letter to represent your unknown. When we are looking for a number we often use the letter n, but you can use x if you prefer. You know that you can get a percentage of a number by multiplying, so just use the same strategy:

$15\% \cdot n = 90$

$0.15n = 90$

n = $\frac{90}{0.15}$

n = 600

With three different types of percentage problems that can be worded in various ways, you may find things a bit confusing. Proportions can also be used to solve these problems, and they provide one single consistent way to deal with all of them. This method relies on placing the unknown quantity [the part you want to find] in the right spot. Let's use proportions to do all of the problems we just solved. For these examples I picked x for the unknown.

1. If you answer 18 out of 20 questions on a test correctly, what is your percentage grade?

Here you want to find the percent, so put the x in the percent part, $\frac{x}{100}$:

$\frac{18}{20} = \frac{x}{100}$

Cross-multiply to solve this proportion and find x:

20x = 1800

x = 90

$\frac{90}{100}$ = 90%

2. 6 is what percent of 300?

Again we have an unknown percentage, so that is where you put the x:

$\frac{x}{100} = \frac{6}{300}$ [or $\frac{6}{300} = \frac{x}{100}$ if you prefer it the other way around]

Cross-multiply to solve this proportion and find x:

300x = 600

x = $\frac{600}{300}$

$x = 2$

$\frac{2}{100} = 2\%$

3. Find 20% of 200.

We know the percentage, but we are looking for some part of the number 200. We expect that number to be smaller than 200, so put the x on top and 200 on the bottom:

$\frac{20}{100} = \frac{x}{200}$

$100x = 4000$

$x = \frac{4000}{100}$

$x = 40$

4. 15% of what number equals 90?

The percentage is known. We need some number that is going to be larger than 90. Put 90 on top and x on the bottom, so that x corresponds to 100:

$\frac{15}{100} = \frac{90}{x}$

$15x = 9000$

$x = \frac{9000}{15}$

$x = 600$

Meeting Up Problems

The "meeting up" problem has fallen out of favor in recent years, maybe because many students find it quite difficult. In this chapter we will continue the problem that has you meeting up with Alyssa, the friend who lives within easy walking distance of your house. You and Alyssa will both leave your houses at the same time and walk towards each other. This means that you will eventually meet. The key to a "meeting up" problem is to pick an unknown but defined distance *from a particular place*. For this one we will select to measure the distance from your house. When Alyssa walks towards your house, you can figure out how far from your house she is at any given time by creating a formula. If you leave your house at the same time, you can also figure out how far from your house you are at that time. The two of you will meet up when you are both at the exact same distance from your house. Give it a try in the quiz.

Chapter 8 Quiz

1. The National Registry of Bad Habits reports its findings in word problems. After all, laziness is a bad habit and should be avoided at all times.

Here is the latest report:

Last month, Reginald picked his nose 5 times less than twice as often as Sue–Anne did. If Sue–Anne picked her nose 12 times last month, how often did Reginald pick his nose? ______ times.

2. Previously, we met Paul and Mike, who started washing cars together. Paul was not as fast as Mike, but now, with a few hints from Mike, he can wash a car in only 25 minutes. Mike still takes 20 minutes. How long does it take them to wash a car when they work together?

Remember that we must add rates, not times. Let's do a per minute rate for this one: Paul and Mike can wash cars together at a rate of ______ cars/minute.

Working at this same rate, how long does it take them to wash one car? ______ minutes [round your answer to the nearest minute]

3. Solve the following problem by using a proportion:

5% of what number is 40?

4. A major automobile manufacturer is getting reports of the brake lights not working on its latest model car. The problem is due to a defective part that costs only $10 to replace. Company experts check random cars and find that 16% of them have the defective part. The company immediately plans a major recall of the vehicle. 60,500 of these cars have been sold so far. Assume that they all come in for a checkup, and only actual defective parts are replaced. How much money should be budgeted for the cost of the replacement parts?

$ __________

5. Here are some problems from the Rhind Papyrus, around 1650 BC:

a. A quantity with 1/2 of it added to it becomes 16; what is the quantity?

____

b. A quantity with 1/5 of it added to it becomes 21; what is the quantity?

____

6. The sum of two numbers is 87. The larger number is 3 less than 5 times the smaller number. What are the two numbers? ________ and ________

7. You will recall that in the quiz for Chapter 5, your friend Alyssa was walking towards your house at a speed of 2.4 miles per hour, and you were walking toward her at a speed of 1.8 miles per hour. Your house is exactly 1 mile away from Alyssa's house. Mathematically speaking, you and Alyssa will meet up when you are both at the same distance d from your house. You found an expression for the distance d that Alyssa is from your house, and also one for the distance d that you are from your house. The meeting occurs when both expressions are equal to each other. Write the proper equation and solve for the only remaining unknown m.

How many minutes after leaving your house do you meet Alyssa? ________ minutes. [Here it makes sense to round your answer to the nearest minute, but since you have more calculations to do you should keep at least one, and preferably two more digits after the decimal point to avoid rounding errors.]

At what distance from your house does this meeting occur? ________ miles. [round your answer to the nearest one hundredths of a mile]

Portfolio Chapter 8

How do you feel about word problems? What are some of your personal strategies for solving them?

Help for Quiz

Quiz 8, problem 1

To solve this somewhat unappealing problem, first consider what the equation would look like if Reginald picked his nose twice as often as Sue–Anne. Not only would that make it easy to guess the answer, but we would simply write it as r = 2s. Next, we fix the equation so that Reginald picked his nose 5 fewer times. Once you have the answer, make sure that it makes sense when you read back over the problem.

Quiz 8, problem 2

If Paul can wash a car in 25 minutes, how many cars can he wash per minute? Well obviously that would be less than one car. In one minute he can only wash a fraction of a car. 1 car/25 minutes = 0.___ cars/1 minute. We can either use a calculator to handle the part on the left side of the equation, or we can treat the equation as a proportion:

$\frac{1}{25} = \frac{c}{1}$ where c is the number of cars.

Cross–multiplying, we get 25c = 1, or c = 1/25. Convert this to a decimal.

Mike takes 20 minutes per car, and we can find his number of cars per minute the same way as we did for Paul.

When we add these rates we get _____ cars/minute. Now we have created our very own conversion factor. Remember that sometimes you have to turn your conversion factors upside down. In this case we need to start with 1 car and have "cars" cancel out so that we are left with "minutes". That will be the answer we want.

1 car x ____ cars/min or 1 car x 1 min/____ cars ?

Only one of these multiplications causes "cars" to cancel out.

Alternatively, you can simplify the problem so you can see what is happening. If Mike and Paul could wash 0.5 cars in a minute, it would take them 2 minutes to wash a one whole car (not likely, but it does look simpler). If they could wash 0.25 cars per minute it would take them 4 minutes to wash a car. If they could wash 0.01, or one hundredths, of a car per minute it would take them 100 minutes to do a whole car.

Quiz 8, problem 4

Proportion questions may involve a percentage rather than a fraction, like this one:

A major automobile manufacturer is getting reports of the brake lights not working on its latest model car. The problem is due to a defective part that costs only $10 to replace. Company experts check random cars and find that 16% of them have the defective part. The company immediately plans a major recall of the vehicle. 60,500 of these cars have been sold so far. Assume they all come in for a checkup, and only actual defective parts are replaced. Approximately how much should be budgeted for the cost of the replacement parts? Round your answer to the nearest dollar.

Even if we understand proportions, we can get confused when the problem contains a percentage instead of a fraction. However, you can just convert the percentage to a fraction to make it easier. 16% is just 16 out of 100. If 16 out of 100 cars have the defective part, how many out of 60,500 will have that problem? Just set it up as a proportion:

$$\frac{16}{100} = \frac{x}{60{,}500}$$

Now you should be able to solve for x, the total number of cars with the defective part. Alternatively, you could just find 16% of 60,500 directly (16% is 16/100ths, or 0.16. We can simply do 0.16 times 60,500, which will give us 16% of 60,500.) Once you know how many cars need a replacement part, remember that the cost is $10 for each car.

Quiz 8, problem 5

Instead of "a quantity", you can use a simple unknown, like x. Then the problem reads $x + \frac{1}{2}x = 16$. You can solve this problem in two ways. The first way involves adding 1x and $\frac{1}{2}x$ together. The other way gets rid of the fraction immediately by multiplying both sides of the equation by 2.

Quiz 8, problem 7

d = “expression 1” and d = “expression 2”. When the two distances are equal, “expression 1” = “expression 2”. Your remaining equation will have only one unknown. After you get your answer, look at it to see if it makes sense.

m = _________ minutes

Once you know when the meeting takes place, you can use either d = “expression 1” or d = “expression 2” to find the distance from your house. You should get the same result using either one of these equations.

d = __________ miles

Test 1

Review for Test 1

You are about to take the first test for this course. A good way to study for this test is to create a portfolio summary page. Even if you don't need portfolio pages, you should make them anyway. After this course, an entire year will go by while you take Geometry. When you get to Algebra II, it is very useful to have a summary to refresh your memory. Also, just making the pages will help you remember things better.

Suggestions for creating portfolio pages to help you study:

Write down each of your algebra tools and skills, and show yourself solving a challenging problem using that skill. Include all the steps you took to solve the problem, and any explanations you think might help you remember what you did a year from now. You can get the problems from quizzes, practice web links, or by making up your own.

Algebra Tools:

- Using a letter to replace an unknown quantity
- Algebra notation, including omitting the multiplication sign, and using $\cdot$, $\geq$, $>$, $<$, and $\leq$
- Manipulating negative numbers, and the absolute value sign $|x|$
- Understanding that 3a means a + a + a
- Temporarily replacing an unknown with a number so that you can work with it more easily, or check your work.
- The distributive property: 5 (a + b) = 5a + 5b
- Working backwards from the distributive property (also called factoring)
- Understanding that $\frac{3a}{3} = a$ and $\frac{3a}{a} = 3$
- Rearranging $\frac{a}{b} = c$ into $\frac{a}{c} = b$ or $a = bc$

New Skills:

- Balancing equations
- Taking a shortcut by cross–multiplying

- Solving inequalities
- Setting up and solving proportions

Test 1

1. $-81 \div -9 < -81 \div 9$

Answer: _____ True or _____ False

2. p is a positive number. $2p < 3p$

Answer: _____ True or _____ False

3. n is a negative number. $2n < 3n$

Answer: _____ True or _____ False

4. Divide: $\frac{-6x^2}{3x} =$

5. Divide: $\frac{5x+5}{5} =$

6. Use the distributive property: 3x (x – 1) =

7. $\frac{x+9}{5} = x - 3$ x =

8. $\frac{a}{b} \cdot \frac{1}{5} =$

9. $\frac{1}{a} \div \frac{1}{b} =$

10. A snail wakes up in the morning, and decides to head back to the puddle where it had a drink the night before. The edge of the puddle is 1.27 meters away. The snail starts moving directly toward the puddle at a steady speed. After 3 minutes, it has traveled 12 inches. What is the snail's speed in inches per minute? ______ inches /min. At this speed, how long will it take for the snail to reach the puddle? ______ minutes.

[Conversion factors: 1 inch = 2.54 cm 1 meter = 100 cm]

11. x + 2 = y. |x – y| + |y – x| =

a. 0

b. 4

c. –4

d. 2

12. A rectangle has a perimeter of 58 cm. The length of the rectangle is 5cm longer than the width. The width of this rectangle is ______ cm.

13. Add: (8a + 5b) + (3a – 2b) =

14. Subtract: (5x + 4y) – (3x – 2y) =

15. Multiply: $x(3x + 5 - y) =$

16. $-x + \frac{1}{5} = \frac{4}{5}x + 2$ $x =$

17. For all positive and negative numbers a and b, $|a - b| = |b - a|$

Answer: ____ True or ____ False

18. $2x + 2 < 10$ $x <$ _________

19. $4x + 7 - 5x < 2$ Write the correct expression for x: ___________________

20. $\frac{x}{a} + \frac{3}{b} =$

21. If $x/y = 4$ and $y = 3$, then $xy =$

a. 4
b. 9
c. 12
d. 36

22. If $a - b = b - 4 = 5$, then $a =$

a. −3
b. 6
c. 9
d. 14
e. 18

23. If $6x - 1 = 3a$, then $(6x - 1)/3 =$

a. a/3
b. a/6
c. a
d. 3a

24. If $4x - 2(x + 6) < 5x - 42$, then

a. $x > 54/3$
b. $x > 10$
c. $x < 10$
d. $x > 16$

25. Joanne drives at an average speed of 60 miles per hour for 1 hour. How long would Jim have to drive at an average speed of 30 miles per hour to cover the same distance?

a. 20 minutes
b. 1/2 hour
c. 1 hour
d. 2 hours
e. 3 hours

26. The number x lies within 3 units of 10 on a number line. We could describe this relationship as

a. $|x + 10| \leq 3$
b. $|x - 10| \leq 3$
c. $|x - 3| \leq 10$
d. $|10 - 3| \leq x$
e. $|x + 3| \leq 10$

27. A 10 ounce bottle of shampoo costs x dollars, while a 20 ounce bottle of the same shampoo costs y dollars. If the larger bottle costs z dollars less than 2 of the 10 ounce bottles together, which of the following equations must be true?

a. $x - y - z = 0$
b. $x - 2y + z = 0$
c. $2x - y + z = 0$
d. $2x - y - z = 0$
e. $x + 2y - z = 0$

28. An engineer needs to know the weight of the screws he intends to use in a project. If 7 screws and a 10 gram weight balance on a scale with 3 screws and a 30 gram weight, what is the weight of one screw?

a. 2 grams
b. 4 grams
c. 5 grams
d. 7 grams
e. 10 grams

29. A machine at a factory can produce fan blades at a rate of 120 per hour. An employee can package these blades at a rate of 3 per minute. How many employees are needed to keep up with 18 machines of this kind?

a. 10
b. 12
c. 16
d. 22
e. 24

30. Three different integers have a sum that is exactly equal to their product. The average of these three integers is also equal to their sum. If two of the integers are 0 and −15, the third integer is

[Note: the word integer means a positive or negative whole number, or zero.]

a. 15
b. 0

c. 2
d. 10
e. −15

31. If 44 · 2 · p = 8, then p =

a. 10
b. 11
c. 20
d. 1/10
e. 1/11

32. If m < n < 0, which of the following is the largest quantity?

a. m
b. −m
c. 0
d. −(m + n)
e. −n

Assignment: Does This Algebra Stuff Really Work?

Consider the following case, where an unknown number a is equal to an unknown number b:

$a = b$

$a^2 = ab$

$a^2 + a^2 = ab + a^2$ [add a^2 to both sides]

$2a^2 - 2ab = ab + a^2 - 2ab$ [subtract 2ab from both sides]

$2a^2 - 2ab = a^2 - ab$

$2(a^2 - ab) = 1(a^2 - ab)$

$2 = 1$

Carefully check each step. Is there something wrong with this, or is 2 really equal to 1??

Chapter 9: Equations with Two Unknowns

Okay, enough of the basics. How about looking at some methods that solve problems that would be very hard to tackle otherwise. These are usually problems that involve two unknown quantities. Here is our first one:

Keesha has some apples and some oranges. She has 12 fruits altogether. How many apples and how many oranges does she have?

Right away, you can see that something is missing. There are two unknown amounts, but to solve this problem you need more than one piece of information about these unknowns. If we call the number of apples a, and the number of oranges o, all we can say about this problem is that $a + o = 12$. Without more information we cannot find a or o.

A simple example of a problem with two unknowns and two pieces of information about them is the following:

Tom has \$1.15 in nickels and dimes. He has 15 coins altogether. How many nickels does he have? Now, if Tom knows that he has \$1.15 there should be no reason why he can't just count his nickels, but real applications of this type of problem might involve subatomic particles, isotopes, nutrients in cattle feed, and many other things that are kept out of basic algebra textbooks so as not to confuse students. Getting back to Tom, we need to name the unknown quantities. I'll use d for dimes and n for nickels just to be original. Because there are 15 coins, I know that $d + n = 15$. The next thing we know is that the total amount of money is \$1.15. To avoid nasty decimal numbers, we can use cents for this problem rather than dollars, so the total amount will be 115 cents. Each dime contributes 10 cents to the total, and each nickel adds 5 cents.

We also say that two dimes would contribute 10 cents times 2, and 2 nickels would contribute 5 cents times 2.

In general, d dimes would contribute 10 times d cents to the total, and n nickels contribute 5 times n. Therefore $10d + 5n = 115$. The two pieces of information we have gathered have resulted in two different equations, each with two unknowns:

$d + n = 15$
$10d + 5n = 115$

But now what? At first glance it may seem like we're stuck at this point. Fortunately, other people have already done all the hard thinking for us, and we can just use an established method: substitution. $10d + 5n = 115$ would indeed be difficult to solve, except for the fact that I know something about d, which is that $d = 15 - n$.

Using this knowledge, I can change my equation into something that doesn't mention d at all:

$10(15 - n) + 5n = 115$

Notice that d has been carefully replaced by $15 - n$. When I say carefully, I mean that I have used parentheses so that the entire amount $15 - n$ will be multiplied by 10. **Always use parentheses when you make your substitution!** Now there is only one unknown amount, which is n.

Using the distributive property, I write
$10\cdot 15 - 10n + 5n = 115$, or $150 - 10n + 5n = 115$.

Add -10 n and 5n together to get $-5n$.

$150 - 5n = 115$

Removing 150 from both sides of the equation leaves $-5n = -35$.

Whenever you end up with a − sign on both sides of an equation, you can divide (or multiply) both sides by −1 to quickly get rid of it:

$5n = 35$

That makes it obvious that you should divide both sides by 5 to get $n = 7$. This means that Tom has 7 nickels. Don't take my word for it; add things up and check that the answer is correct.

Adding Equations

The substitution method is not the only way to handle a problem that contains two equations. As we saw before, you can do almost anything with an equation so long as you don't change the balance.

Let's go back to when we added elephants to some very large balance scales. Recall that we added an identical elephant to both sides. You might have imagined that they were identical twins, but in fact elephant twins are very rare. Identical twins would be even more rare, and even so they wouldn't weigh exactly the same. Imagining things in a more realistic way, we'll say that we have only one elephant that conveniently weighs exactly 10,000 pounds. We can balance that out with 10,000 pounds of rocks, like this:

Elephant = 10,000 pounds of rocks

The original equation was

$x + 2 = 7$

Imagine x and 2 on one side of some very large balance scales, and 7 on the other. Adding the elephant on the left and the rocks on the right, you can see that the scales still balance. In general, we can add equations like this:

$$\begin{array}{lcl} \text{Elephant} & = & \text{10,000 pounds of rocks} \\ \underline{x + 2} & \underline{=} & \underline{7} \\ \text{Elephant} + x + 2 & = & \text{10,000 pounds of rocks} + 7 \end{array}$$

Make sure to carefully line up the equals signs so you don't accidentally add something to the wrong side.

In the same way, you can take one equation and add another one that refers to the same problem. That may sound strange, but remember that the second equation also has two equal parts. Say we decide to take the first equation and add the second one. That means we take the part on the left side of the first equation, and add the part on the left side of the second equation. Then we take the part on the right, and add the right half of the second equation. Because we have added an equal quantity on both sides, the equation is still valid. [While you could take a random equation from different problem and add it, that would get you nowhere. The unknowns and their values have to match so that x + - x would be 0. You can't take a different x from another problem and expect to get a sensible answer.]

Using some actual real equations, it looks like this:

$$\begin{array}{l} 5x - 6 = 9 \\ \underline{3x + 4 = 13} \\ 8x - 2 = 22 \end{array}$$

It works, provided both equations are referencing the same x. Here I used x = 3 to set up both equations, and if you solve the sum equation you should find that x is still equal to 3.

In the same way, you can subtract equations. This is a bit more likely to cause mistakes because it is hard to remember to subtract each term when there are minus signs involved. If you choose to do a subtraction you need to use parentheses:

$$\begin{array}{rl} & 5x - 6 = 9 \\ - & \underline{(3x + 4 = 13)} \\ & 2x - 10 = -4 \end{array}$$

A safer way to do this is to multiply the second equation by -1 first, changing all the signs. Then you can add, which is just easier than subtracting.

The most easily avoidable mistake when adding equations is to not line up your equals signs so you end up accidentally adding something to the wrong side. I had to spend a lot of time adjusting tabs in this document to get those signs exactly right – please take just a second and do it correctly on your paper.

Going back to our two equations about nickels and dimes, we have

$d + n = 15$ and $10d + 5n = 115$.

Adding the two equations is possible, but that would do nothing to help us. However, if the first equation contained either -10d or -5n, something would cancel out when you add. A multiplication by either -10 or -5 would put the first equation into a more convenient form. I picked -10:

$$-10(d + n = 15)$$

$-10d - 10n = -150$

Now it becomes helpful to add:

$$\begin{array}{rcl} -10d - 10n & = & -150 \\ \underline{10d + 5n} & \underline{=} & \underline{115} \\ -5n & = & -35 \end{array}$$

The unknown d magically disappears. Dividing both sides of the equation by -5 shows us that $n = 7$, again.

You probably feel that the result wouldn't be any different if I had chosen to multiply the first equation by -5 to eliminate the number of nickels, and then figured it out from knowing how many dimes there were. Just the same, you should grab a piece of paper and check that out yourself. Solving equations with two unknowns is an important skill that takes practice to learn and remember.

Little Plastic Models Inc. is producing bags that contain 100 toys. Some of the toys are little plastic soldiers weighing exactly 1 gram each, and some are little jeeps for the soldiers to ride around in. The jeeps weigh 3 grams each. After the first batch of bags has been produced, a quality control employee checks a random bag and finds that it weighs 120 grams. Assuming that the bag does in fact contain 100 toys, how many of each kind are there?

Let's use s for the number of soldiers, and j for the number of jeeps. We know that there are 100 toys in this bag so $s + j = 100$. Every little soldier adds 1 gram to the total weight, so s soldiers add $1 \cdot s$ which is s grams. The jeeps add 3 grams per jeep, and there are j of them, so that's 3j grams. The total weight of 120 grams is made up of s grams + 3j grams.

$s + j = 100$

$s + 3j = 120$

It is easy to get rid of s, because we know that $s = 100 - j$. Substituting that into our second equation, we get $100 - j + 3j = 120$. We simplify that to $100 + 2j = 120$.

Subtracting 100 from both sides, we are left with $2j = 20$, or $j = 10$. That's 10 jeeps, which means there must be 90 little soldiers, since s + j = 100.

Overall, I prefer the substitution method, but many problems are constructed to favor adding equations.

In this fairly simple case I would probably get lazy and subtract, putting the smaller numbers on the bottom to make it easier:

$$\begin{array}{r} s + 3j = 120 \\ \underline{-(s + j \;\; = 100)} \\ 2j = -20 \end{array}$$

$j = 10$

Then I would carefully check my answer to make sure I haven't made a mistake. A safer way is to multiply by -1 and add: -1(s + j = 100):

$$\begin{array}{r} s + 3j \;=\; 120 \\ \underline{-s - \;j \;\; = -100} \\ 2j \;=\; 20 \end{array}$$

Again, $j = 10$

Next we'll tackle a more challenging situation that involves some chemistry. Argon gas normally contains only about 0.3% of a lighter argon atom called argon 36, while most of it is made up of heavier argon 40 atoms. A scientist is trying to create a sample of argon gas that contains at least 50% of the lighter atoms. He has determined that he has 1.000 moles of atoms in his sample, which is 602300000000000000000000 atoms (hmm, it's really no surprise that chemists would count their atoms in weird units called moles). The sample weighs 38.80 grams. 1 mole of argon 36 weighs 35.97grams, and argon 40 weighs 39.96 grams per mole. How many moles of argon 36 does the sample contain?

Immediately we notice that even though I have simplified this problem considerably from real life, it looks very complicated. It has a lot of icky decimals, and instead of dealing with a familiar unit like dollars we now have to work with something called a mole. This is the kind of problem that strikes fear into the hearts of many college students, and they seem to forget that the basic algebra they learned in high school is still there to help them.

First, "the number of moles of argon 36" is still a simple unknown quantity. I'll call it t for thirty–six. The other unknown is "the number of moles of argon 40", so I call that f for forty. Looking back at the problem, we see that the total number of atoms in the sample is 1.000 moles, so

$t + f = 1.000$

The other thing we know about the sample is that it weighs 38.80 grams. What contributes to this weight? If the whole sample was made up of argon 36 it would weigh 35.97 grams, but it is heavier than that. How about if there was the required 50%, or half a mole of argon 36? That would contribute 0.5 times 35.97grams. We know the actual amount is 0.something times 35.97 grams, and we can treat it like any other unknown: The weight contributed by argon 36 is t times 35.97, or 35.97t. Argon 40 also contributes to the weight: 39.96 times however many moles there are of this kind of atom, which is 39.96f. The total weight of 38.80 grams is made up of argon 36's contribution and argon 40's contribution, so

$35.97t + 39.96f = 38.80$

Along with t + f = 1.000 that makes for a simple algebra problem with two equations:

$t + f = 1.000$

$35.97t + 39.96f = 38.80$

We are looking for the value of t, so let's get rid of f: $f = 1.000 - t$. Plugging that into the second equation we get

$35.97t + 39.96(1.000 - t) = 38.80$

Now we use the famous distributive property:

$35.97t + 39.96 - 39.96t = 38.80$

or 35.97t – 39.96t + 39.96 = 38.80. Yes, we can subtract 39.96t from 35.97t just as easily as we can subtract $40 from $30 by borrowing $10 from someone [just tell your parents it's for educational purposes]. The answer to that is –3.990t.

Now our equation reads

$-3.990t + 39.96 = 38.80$

Subtracting 39.96 from both sides gives us $-3.990t = -1.160$. Here is where I grab a calculator to divide both sides by 3.990 to get $-t = -0.2907$ or $t = 0.2907$. This tells us that 0.2907 moles of argon–36 are present in the scientist's 1 mole sample, or 29.07% which is less than the 50% he is trying to achieve.

In the quiz for this chapter, you'll find a variety of problems that contain two unknowns, along with two pieces of information about these unknowns. The challenge is to construct two equations, and then solve them using substitution, addition or subtraction, or by setting two expressions equal to each other as you learned to do for solving "meeting–up" problems.

Significant Figures

There is one thing you have to be careful of when a chemist, or more likely a chemistry student, asks you for help. To chemists and other scientists, zeros that come after the decimal point mean a lot more than they do to mathematicians. When a chemist says that his sample weighs 38.80 grams, he doesn't mean 38.8. That last zero is the result of a careful measurement that indicates the precision of the result. This zero is just as significant as the other digits, so don't lose it! When we hand the answer back to the chemist it should be just as precise as his measurements, so we have to understand which part of a number is "significant".

You have already learned to use conversion factors, so let's take the measurement 38.80 grams and convert it to kilograms. To do that, we have to know (or look up) that there are 1000 grams in a kilogram. We multiply 38.80 grams $\cdot \frac{1\text{ kg}}{1000\text{grams}}$. The unit "grams" cancels out and we are left with $\frac{38.80\text{ kg}}{1000}$ = 0.03880 kilograms. There are now more digits in our measured number, but converting the measurement to a different unit doesn't make it more precise. The two zeros in front of the 3 are really just "placeholder zeros" that will not be there if we switch back to grams. We say that 38.80 has 4 significant digits, and so does 0.03880.

Zeroes that are somewhere in the middle of a number are obviously significant, so 705 has 3 significant digits. The number 12001 has 5 significant digits. Unfortunately however, when there are zeros at the end of a number we sometimes have to guess what kind of measurement was actually made. For example, you sometimes hear people say that the speed of light is 300,000 km per second. That is an amazingly convenient figure. Does it mean that the speed was measured exactly to the nearest kilometer, and found to be exactly 300,000 km/sec, or was it rounded off? Actually it was, and there is a far more precise figure available. To the nearest kilometer, the speed of light is 299,792 km/sec. That gives the speed of light correct to 6 significant figures. Measuring to the nearest meter gives 299 792 458 meter/sec. This last measurement has 9 significant figures. If you want to show that you have measured something to be precisely 2000 kilometers, you can indicate that by writing 2000. kilometers. The decimal point indicates that the last zero is significant, so your number has 4 significant figures. Some textbooks will place a line under or over ambiguous zeroes to indicate that they are significant. The best way to do things though is to use scientific notation. "Exactly 2000 kilometers" can be written as 2.000×10^3 km. This clearly shows that the measurement was not rounded off and all those zeroes matter. We will see more about scientific notation a few chapters from now, when we tackle exponents.

When you start doing calculations, you may have a problem if one value was measured much more precisely than another. Suppose a developer wants to buy two adjacent properties, belonging to neighbors Jack and Sam. Jack is a very organized person who has kept all documents about his property. He reports that the area of his land is 8.35 acres. Sam can't find his papers anywhere, but he remembers that he has about 10 acres of land. It would not make much sense to say that buying both properties would give the developer exactly 18.35 acres of land. Sam's figure is only correct to the nearest whole acre. **The best we can do when adding or subtracting is to use the lowest available correct place value.** We report the total to the nearest whole acre, or 18 acres. If Sam had said that he has 10.1 acres of land, we would have our information correct to the nearest tenth of an acre. In that case we would report the total as 18.5 acres (10.1 + 8.35 = 18.45, which is 18.5 when rounded to the nearest tenth).

When you are multiplying or dividing, keep as many significant figures as practical during your calculations to avoid rounding errors. However, **the final result of multiplication**

and division can have only as many significant figures as the number with the least amount of significant figures that you used as a basis for your calculations.
For example,
16.54 ÷ 19.00 x 2.7 = 2.4 Both 16.54 and 19.00 have four significant figures, but 2.7 only has two. The final result must be rounded off to two significant figures.

Practice for Equations with Two Unknowns

When you are not doing word problems, the equations are already supplied for you. As we have seen, you can solve for two unknowns if you have two suitable equations. Look online for practice problems for equations with two variables. Try solving these problems using the substitution method explained in this chapter, and also by adding or subtracting the two equations. Do you get the same answers? Plug your answers back into both equations to see if you are right.

Mixture Problems

These types of problems involve adding two mixtures (usually solutions) with different concentrations of a particular substance to create a new mixture with a different concentration. One of these mixtures may have a 0% concentration of the substance, such as pure water which can be added to a solution to dilute it.

Let's consider solution A with a 12% concentration of salt, and solution B with a 3% concentration. We want to add them together to make a solution with, say, 5% salt. Now, percentage measurements can be confusing, because they may refer to a percent by weight, a percent by volume, or even a mixture of the two. Solids dissolved in water are normally measured as a percentage by weight, and in science problems that weight is usually measured in grams, or kilograms (1000 grams) for larger amounts. A 5% salt solution would contain 5 grams of salt for every 100 grams of solution, or 5 kilograms for every 100 kilograms of solution.

Whatever unit we decide to use, as long as we keep it the same throughout it can be omitted from our algebra.

Consider the actual amount of salt as you set this up. Let's call the amount of the 12% solution A. So how much salt is actually in that? We don't have a number, but thanks to the magic of algebra we can say that it is .12 times A. Let's call the amount of the 3% solution B. There is 0.3B salt in that. When you combine the two solutions, you have a total amount of salt given by .12A + 0.3B. We will use that to make our final 5% solution. The amount of that solution is A + B. How much salt will be in it? Well, that will be 0.05 times (A + B). Although salt dissolves, it doesn't magically disappear or appear, so in the end we have the same amount that we started with:

$.12A + .03B = .05(A + B)$

Because we may find it confusing to work with decimals, we can multiply both sides of this equation by 100 to get

$12A + 3B = 5(A + B)$

If we know the total amount of the new solution desired, we can create a second equation. For example, if we want 9 kilograms of the 5% solution, then

$A + B = 9$

Now we can substitute by creating an expression for either A or B. I'll choose to get rid of A, by saying that $A = 9 - B$. Substituting that into the first equation I get

$12(9 - B) + 3B = 5(9 - B + B)$, which works out to

$108 - 12B + 3B = 45$

$108 - 9B = 45$

$63 = 9B$, so $B = 7$. Plugging that into A + B = 9 we get $A = 2$.

Once all the calculations are done we can add the unit back in. If that was kilograms then the amount of solution A should be 2 kilograms, and the amount of solution be should be 7 kilograms.

If the total amount of the new solution is not specified, we can still figure out how to make it.

$12A + 3B = 5(A + B)$

$12A + 3B = 5A + 5B$

$12A - 5A = 5B - 3B$

$7A = 2B$

$A = \frac{2B}{7}$ Now divide both sides by B:

$\frac{A}{B} = \frac{2}{7}$. This means we can take 2 parts (by weight) of solution A and 7 parts (by weight) of solution B to get the desired result. I actually did this first before deciding we should make 9 kilograms of the new solution, so we'd get nice whole numbers for the answer.

Sometimes you may see a problem where one of the solutions has a 0% concentration. No need to panic; it can be solved the same way. Let's work with something more exciting than salt, like maybe concentrated acid. Suppose you have a 40% solution of hydrochloric acid, HCl. This extremely dangerous liquid has to be stored and handled very carefully. Let's add it to some water to dilute it to a more manageable 15%. Even for that you need gloves, safety goggles and long sleeves. Always add acid to water, not the other way around. Even a single drop can splash back at you and cause a nasty burn.

[When water is added to sulfuric acid, so much heat is released that the solution can boil violently, - YouTube]

We'll make 40 ounces by weight of the 15% solution. Using the letter a for the weight of the 40% acid solution, and w for the weight of the plain water, we get two equations:

$a + w = 40$

$0.4a + 0w = .15(a + w)$

Although this is not hard to solve when you leave the decimals, you may prefer to get rid of them by multiplying both sides of the second equation by 100:

$a + w = 40$

$40a = 15(a + w)$

Since w is already gone on the left side of the second equation it is the easiest unknown to eliminate. Rewrite the first equation as $w = 40 - a$. Then substitute that into the second equation:

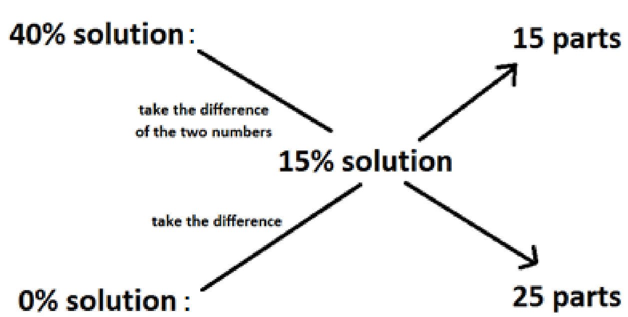
$40a = 15(a + 40 - a)$

$40a = 15(40)$

$40a = 600$

So, a = 600/40 = 60/4 = 15. If a is 15 ounces then w turns out to be 25 ounces because we said that w = 40 – a.

These kinds problems provide a good illustration of the usefulness of algebra, but actually creating the needed equations can be quite challenging. Here is an interesting shortcut used by pharmacists, who have to deal with this stuff a lot for medicines:



This tells you the ratio of the two solutions. 15 parts to 25 parts is the same as 3 parts to 5 parts. To make any amount of a 15% solution, take 3 parts of the 40% acid solution, and add it to 5 parts water, very carefully!

Chapter 9 Quiz

If you have difficulty with any of these questions, be sure to review “How to Solve a Math Problem in 10 Relatively Easy Steps” located near the beginning of the course outline.

1. Marcia and Bob are both police officers. Marcia wants to impress her supervisor and she writes a lot of traffic tickets. Bob is rather easy–going and tends to let people off with a warning. By the end of the year, Marcia has handed out four times as many tickets as Bob. Together they have issued 1260 tickets for the year. How many tickets did Bob issue?

2. Solve the system of equations (find the values of x and y).

$$\frac{x}{y} = 5$$

$$3(y + 2) = 15$$

3. A string that is 112 inches long is cut into three unequal parts. The longest part is 4 times as long as the shortest piece, and the middle piece is three times as long as the shortest piece. What is the length of each piece of string?

4. The sum of two numbers is 58. The larger number is 10 more than 3 times the smaller number. What is the larger number?

5. The classical word problem for this topic is where a farmer has some animals like cows and chickens, and he knows how many animals he has altogether, and how many legs they have total, but he doesn't remember how many of each animal he has. This problem could be considered insulting to farmers, so let's clean it up a little.

A farmer has some cows and some chickens. She likes to tell visitors that she has 20 animals, and that they have 56 legs altogether. She then asks visitors to guess how many cows and how many chickens she has. The farmer has _____ cows and _____ chickens.

6. You are working for a catering firm. When you are putting the day's orders into the computer, you notice that you have a check from Mrs. Scheitzel for $1327.50, but no corresponding order form. You go outside and desperately search your car but you can't find it anywhere. Mrs. Scheitzel was going to have a cocktail party, and you remember that she invited 50 guests, because it was a nice round number. She ordered the canapés, and she followed your recommendation of **5 canapés per guest**. Unfortunately some of the guests were children, and children's canapés are \$1.95 each, compared to the \$5.95 your company charges for the adult canapés. You could phone Mrs. Scheitzel and ask her to repeat how many children she is inviting, but while you were at her house taking the order she was looking at you like you were some lower form of life. If you call her your grumpy boss is sure to hear about your incompetence. You're just lucky she paid up front, or you'd be unemployed when no food arrived at her party. Use your algebra skills to reconstruct Mrs. Scheitzel's order.

____ children times 5 times \$1.95 = \$ ________
____ adults times 5 times \$5.95 = \$ ________

7. You have been hired to work on a llama farm for the summer. When you arrive at work, the farmer asks you if you are good at math. Not wanting to make a bad impression on your first day, you say yes. The farmer looks relieved. He takes you to a barn where a large llama stands alone in a stall. "Had pneumonia, this one," the farmer says. "Doc told me to increase the protein in his food to 20%. Now this hay mix has 15% protein, and this here concentrate has 40%. Scale is over there, and you can borrow my calculator. Just mix up 5 pounds of food for this guy, will you? Oh, and don't forget to clean out his stall." The farmer leaves. The llama seems to have caught on instantly that you are now responsible for its next meal, and it is staring at you intently. When no food materializes, it narrows its eyes and snorts threateningly. How much hay and how much concentrate do you need?

If I were you, I'd solve this problem by any means possible. I hear llamas spit at people when they get ticked off. If you get to the answer some other way maybe you can work backwards to find the equations.

8. (Challenge question) No math course would be complete without a problem involving trains meeting up. To make it more interesting, let's put them both on the same track.

You are a controller supervising train traffic. An unusually large solar flare is interfering with communications and causing malfunctions of the switches that direct trains to different tracks. It is 9:15 a.m. An alarm goes off and your screen shows that two trains are on a collision course. Train A is 10 miles east of you, moving west at 50 miles per hour. It is a 0.3 mile–long passenger train carrying the President of the United States, 78 underprivileged Romanian orphans, Dr. Bloomberg, who is a Nobel prize winner carrying the only copy of his unpublished paper outlining the cure for cancer, and a 90 year old mathematician who has just discovered the proof of an unsolved math theorem. Train B is a freight train 8 miles west of you, moving east at 30 miles per hour. There is a manual switch 0.6 miles west of your control center that could divert train A to a different but parallel track. Desperate to save the mathematician, you rush out of the control center and run for the switch. You pull the handle and the switch slides into position at exactly 9:25 a.m. Has the collision been avoided? (Round your answer to the nearest minute.)

a. Yes, the trains start passing each other at ______ a.m.
b. No, the trains collide head-on at ______ a.m.
c. No, the freight train hits the back of the passenger train before it can completely enter the other track. The collision occurs at ______ a.m.
d. It is too hard to predict, let’s just skip to the solution section.

Help for Quiz

Quiz 9, problem 2

Use the principle of substitution. Here it is easiest to find an expression for x from the first equation, and substitute that into the second equation.

Quiz 9, problem 4

You may want to do this in two stages. First consider that the larger number is 3 times the smaller number. How would you write that in an equation? Does your equation work out for some sample numbers, like 27 and 9?

Next, adjust the equation so that the larger number is 10 more than 3 times the smaller number. Check that your equation works for sample numbers like 37 and 9. Once that's done, find the actual numbers by considering that their sum has to be 58. It is fine to find the answer by guessing, but then go back and use the equations to see how it works.

Quiz 9, problem 5

Your first equation is easy to find, because the total number of cows and chickens is 20. Notice that both things start with a c, so pick another letter for one of them. The total number of legs is 56. Each cow contributes 4 to this total, and each chicken contributes 2. Put an equation together and see how it works if there are 10 cows and 10 chickens. That should give you 30 legs. Now use both equations to find the real answer.

There is an alternate way to solve this problem because it is relatively simple. Create 20 generic animals and give them each 2 legs. Now you have used up 40 legs. Take the rest of the legs and put them on some animals two at a time to make cows. When you run out of legs count how many cows you have. The remaining animals are chickens.

Quiz 9, problem 7

This problem may seem quite difficult when you first look at it. It may help you to consider the total amount of protein that you need, and then think how much each type of food contributes to that total. This is an example of a mixture problem.

Quiz 9, problem 8

This word problem is very similar to the one where you meet up with Alyssa. It just looks a little harder and contains more details. We can tackle it the same way, by picking a fixed point and finding expressions for the distance that the trains are from this particular point. Just as before, the trains meet when these distances are equal. We set the two expressions equal to each other and solve for the only remaining unknown, which is the time.

There are three “locations” available in the story to pick as a fixed point. The control center is in the center of the action and would actually be the most confusing since the trains are on either side of it. I am going to pick a point similar to “your house” in the problem with Alyssa, which will be the location of train B at the start of the problem (but I could just as easily have picked the location of train A). I am also going to assume that when the problem specifies the location of the train it means the front end, since that is the first part to be involved in any potential collision. Afterwards I will check the location of the rear end of the passenger train.

Train B is moving away from my chosen point at a speed of 30 miles per hour. The formula for the distance d that it covers in h hours is d = ____h (for example, in 3 hours it would cover 90 miles). Train A is 18 miles away from my chosen point, moving toward it at a speed of 50 miles per hour. Its distance from my chosen point is given by the formula d = 18 – ____h. When the two distances are equal, ____________________. Solving this I get h = 0.___

Since 0.____ hours is hard to work with I use a conversion factor, 60 minutes per hour, to get ____ minutes. Rounding that to the nearest minute gives ____ minutes. Since it was 9:15 a.m. at the start of the problem, the trains will either collide or start passing each other at ______ a.m.

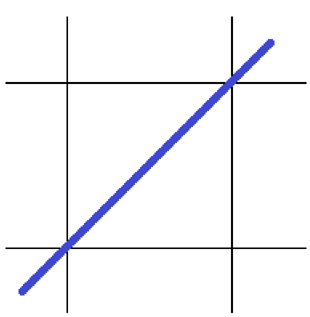
Portfolio Chapter 9

Some of the problems in the quiz are quite challenging. If you solved them you can be proud of yourself and show how you did one or two for your portfolio.

Chapter 10: Caution! Steep Slope Ahead

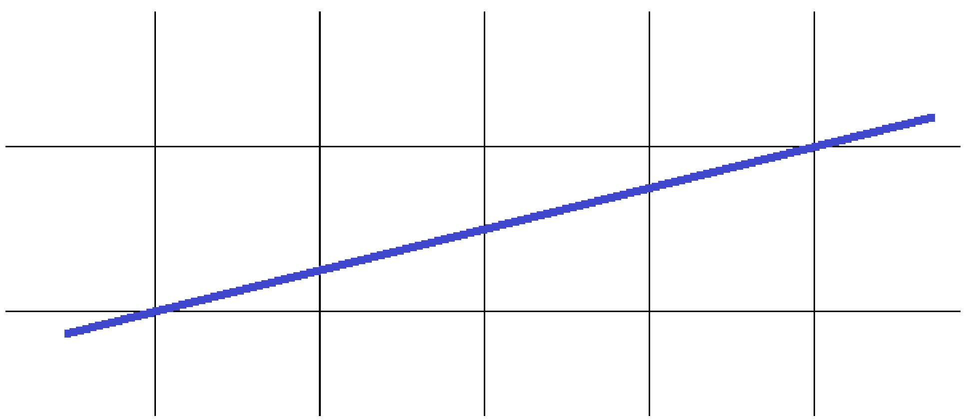
If we have a simple equation with one unknown, we can find out the value of that unknown. Given that $x + 3 = 5$, there is obviously only one value for x. Given that $x^2 = 25$, there are clearly only two values for x [head back to Chapter 2 if you don't know what they are]. Sometimes it can be nearly impossible to find the value[s] of x, as in $189x^{12} + 17x^9 - 75$, but we still see that we could potentially do so, or maybe program some supercomputer to do it for us. Now what about $x + y = 10$? Without any further information, that's it. With two unknowns, we're just stuck. Or are we?

This is the time to take out your graph paper. Notice that it is totally covered with little squares, which allows you to play a game with one other person, preferably with two different color pens. You start by creating a playing field on the paper; just draw a box. Now each person takes turns drawing a line on one edge of a little square. The person who actually closes the square gets to put a dot of their color inside of it. Whoever gets the most squares wins the game. The reason I'm telling you this is that graph paper is usually sold in inconveniently large quantities, and no sane person should draw that many graphs. Anyway, getting back to the subject, you need to pick a random square near the middle of a sheet of graph paper, and using a ruler, draw a line from the lower left corner of the square to the upper right corner. If you are less confused about left and right than I am, you should now have a little line that goes up as you look from left to right.



Take your ruler again and lay it along this small line so you can extend it on both sides. If you do that very carefully you get a line that passes through the diagonal corners of a whole bunch of squares. Notice that when you picked the two corners of a square to start drawing a line, you defined that line. It is not possible to draw a different line using those same two corners. People have noticed that before, so we say that two points define a line, which is just common sense. Once we have a line, we can do stuff with it.

I'm going to take my line, and imagine that it is a hill. Next, I imagine a little guy on a little bicycle trying to ride up that hill. Once I imagine something, it takes off on its own. The little guy turns to me and says, "You've got to be kidding, you don't pay me enough to ride up that big hill." He turns around and rides down the hill instead. Hmmm, I wasn't planning on paying him at all. I try to imagine a bigger guy on a bigger bike, but when I extend the line to make a bigger hill, it's just as steep. Let's create a better hill to ride a bike on. Select 4 adjacent horizontal squares on your paper, a little below your first line. Now carefully draw a line from the left lower corner of the square that is furthest to the left, to the top right corner of the rightmost square. Extend your line through the next group of 4 squares and so on.



If you imagine that this line is a hill, you'll notice that it is not nearly as steep. People have developed ways to measure almost everything, and the steepness of hills is no exception. The standard way to measure the slope of a hill is called "**rise over run**", which means you look at the amount the hill goes up over the horizontal distance that you are measuring. Our first line went up 1 square vertically for every square we moved to the right horizontally. The slope of this line is 1 / 1 which is 1. The second line goes up only 1 square for every 4 horizontal squares. The "rise" is 1 and the "run" is 4, so the slope is 1/4 or 0.25.

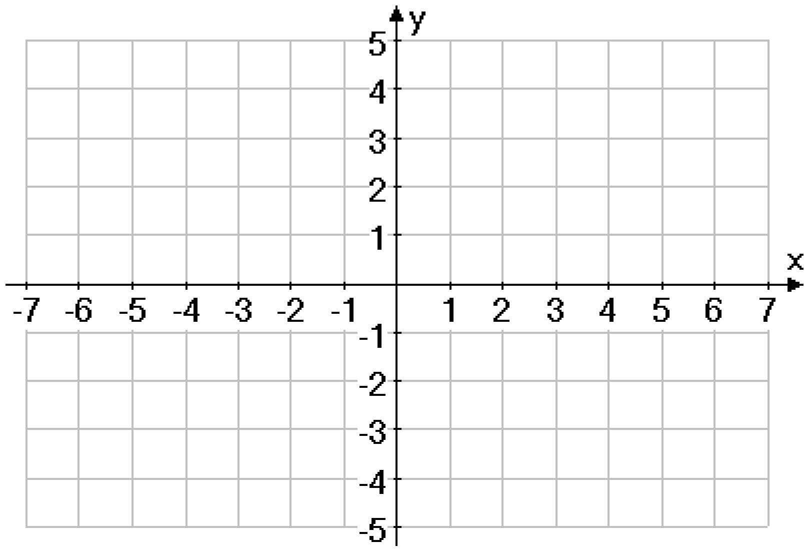
Notice that it makes no difference if we measure over a longer distance, such as 8 squares horizontally. The rise is 2 squares per 8 horizontal squares, so the slope is 2/8 which is still 0.25.

Hilly roads often have warning signs that express the slope as a percentage:

Here the slope, rise over run, is 8%, or $\frac{8}{100}$. That means that the rise is 8 units for every 100 units of run. It doesn't matter if you measure in feet, or yards, or meters, because the ratio is always the same.

By the way, you'll want to memorize the phrase "rise over run" because it is useful sometimes and it takes only a few brain cells to store. Just repeat it a few times and it should stick.

Again I imagine my little guy on the bike to see if he'll ride up the second hill. Unfortunately, he's gotten smarter. He gets off his bike, turns my paper so the line is level, and then rides off happily. One thing we have not provided is a reference system for measuring up and down, and right and left. Let's do that now. Take out a new piece of graph paper and draw a horizontal line near the middle, exactly on the edges of a horizontal row of squares. Then draw a vertical line, again near the middle of your paper, also along the edges of squares. The two lines should intersect somewhere near the middle point of your paper. Each line now becomes a number line, with the 0 point at the intersection. Label both the horizontal and vertical lines with numbers as shown in the example.



Usually labeling from –10 to 10 is enough, and we just imagine the rest or add more numbers as needed. The horizontal line is called the x–axis, and the vertical line is called the y–axis.

Now we have a system that can define every spot on the paper as a specific point. Where the two lines intersect is the **origin**, which is the point (0,0). Each spot on the grid can be found by marking its horizontal (side to side) distance from the origin, followed by its up and down, or vertical distance. The horizontal distance comes first, possibly because we are used to reading from left to right. The horizontal distance is measured along the x–axis. Notice that the numbers are small on the left and get larger on the right. After you determine the horizontal distance, or **x–coordinate**, of a point, you then find the vertical distance. This is measured along the y–axis. The numbers along the y–axis go from small at the bottom to larger towards the top. The y–coordinate tells you where a point is located in the up–and–down direction.

The point (2,1) is found by going to the spot labeled 2 on the x–axis, and then up 1 square, or 1 unit along the y–axis. To find the point (–3,–2), start at the intersection of the two lines (0,0), then go 3 units to the left, and 2 units down. You can also use decimal numbers, such as (0.5,2.3), although it is more difficult to mark such points accurately. This system of defining points is called a Cartesian coordinate system. It is named after the French mathematician and philosopher Rene Descartes who first described it in 1637. The most common problem that students have when using this system is forgetting that the x–coordinate comes first, so it helps to remember that x comes before y in the alphabet.

If you play around with Cartesian coordinates a bit, you'll soon notice that there is one section where both coordinate numbers are always positive. This section is called quadrant I. Moving counter–clockwise, we get to quadrant II. Here the first number, or x–coordinate, is always negative, while the second number, or y–coordinate is always positive. Quadrant III is composed entirely of points having negative coordinates, and in quadrant IV the first number is positive while the second is negative. Knowing these quadrants just means that you understand how the coordinate system works. There is no need to memorize them for the quiz .

In the Cartesian coordinate system, not only do two points define a line, they also define its slope. Draw a line from (0,1) to (6,4). You can extend this line in both directions in only one way, and it always stays at the same angle relative to the x–axis and the y–axis. If you've drawn your line carefully, you can measure its slope anywhere, but let's measure it between the two points I mentioned. Start by putting your pencil at (0,1). To get the "run" or horizontal part, move the point of your pencil 6 units to the right. Because the x–axis is a number line, we are moving in a positive direction by going to the right. For the "rise", or vertical part, notice that your pencil is at 1 relative to the y–axis. Move up to get to 4, which is 3 units. Again notice that we are moving in a positive direction to get the rise. Since the slope is $\frac{\text{rise}}{\text{run}}$, we get $\frac{3}{6} = \frac{1}{2}$.

But what if we start at the other point? Put your pencil at (6,4). Now, to get the "run" you have to move in a negative direction: 6 units to the left. This means that the run is now −6. And the rise.... seems to have turned into a drop. You need to move your pencil down 3 units to get to (0,1). The rise is −3. The slope is $\frac{-3}{-6} = \frac{1}{2}$. This means that if you are calculating the slope of a line from two given points, it doesn't matter which point you start at, which makes sense since the slope should be the same either way.

It is also possible to have a line with a negative slope. To see one, use the points (0,8) and (2,0) to draw a line. Starting at (0,8) the run is 2 units in a positive direction. The rise is a drop, or −8 units. The slope is $\frac{-8}{2}$ which is −4. Check that the slope works out the same if you start at the other point.

Now we'll get some practice and discover a formula for calculating the slope. But wait, we haven't solved the problem that we started this lesson with, x + y = 10, the equation with 2 unknowns. This must be the math equivalent of a cliffhanger. Tune in next week for another exciting episode ....

Coordinate Practice

In the old days, you would use up a lot of graph paper just to learn about coordinates. Now you can do it on your computer screen. Just search online for "coordinate system practice", and you'll find sites with little games that help you get used to using coordinates.

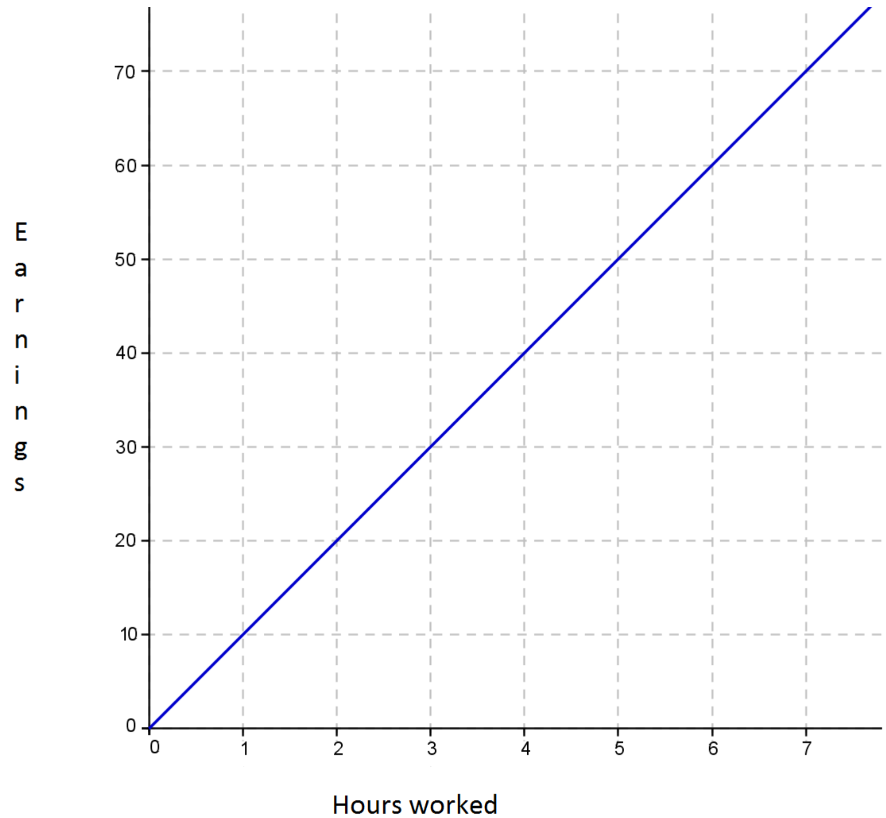
Slope Practice

An online search for "slope practice" will direct you to sites that help you understand slopes. Pay attention to what makes a slope steeper, and what makes it negative. What is the slope of the X–axis? What is the slope of the Y–axis? Make sure you understand why!

For any horizontal line, including the x–axis, the rise is 0, while the run depends on which two points you pick. Zero divided by anything is 0, so the slope of any horizontal line is 0. When you look at a vertical line, you see that the run is zero, while the rise can be anything. However, you can't divide by zero, so vertical lines like the y–axis have an undefined slope.

Real–Life Slopes

People like to make graphs of various real–life situations to help them visualize what the numbers mean. We can make graphs where we show one quantity on the x–axis and one quantity on the y–axis. Here is a really simple example: Amanda's first job was at a local grocery store, where she earned $10 per hour. Amanda's earnings can be calculated by taking the number of hours worked and multiplying by 10. Earnings = 10 · hours worked, which can be abbreviated as $E = 10h$. The relationship between the number of hours she works and the money she earns is shown in the graph below:



First, notice the scale used for this graph. Along the x–axis we have normal units, but along the y–axis each space represents 10 dollars. If we had not modified the scale on the y–axis, the graph would be too tall. Also, only the positive sides of the x–axis and y–axis are shown, because negative values would not make sense in this situation.

Many real life relationships that we can graph will look like a straight line. We say that such relationships are **linear**. To determine the slope of this particular line, you still take the rise and divide it by the run. You should find that the slope is 10, even though the line does not look that steep. The change in scale on the y–axis makes the slope look less steep than it is.

Now think about what the slope means. The rise is $10 for every unit of the run, which represents 1 hour of work. The slope shows Amanda's rate of pay, which was $10/hr.

Rate of Change

When Amanda worked at the grocery store, her money was increasing at a rate of $10 per hour. Now suppose that she comes home and finds that a plumbing leak has flooded her house. While the plumber works to fix the leak, Amanda's money is decreasing by the $90 hourly rate the plumber charges. Plumbing services are expensive, so the total amount of Amanda's money is going down faster now than she can earn it. A **rate of change** tells you how fast something is increasing or decreasing. For a linear graph, all you have to do to find the rate of change is to look at the slope of the graph. When Amanda works at the store, the rate of change of her money is $10 per hour, which is also the slope of the line in the graph we looked at. During the time the plumber works, the rate of change of Amanda's money is – $90 per hour (a negative rate of change).

Chapter 10 Quiz

Use the points in the following questions to draw lines in a Cartesian coordinate system. Determine the slope of each line, and express the result as a positive or negative whole number or simplified fraction.

1. (0,0) and (8,4) Slope =

2. (2,1) and (4,7) Slope =

3. (−3,5) and (3,−7) Slope =

4. (−8,−8) and (−1,−1) Slope =

5. (−1,1) and (5,1) Slope =

Without drawing an actual line, attempt to determine the slope of the line that passes through the points in the following questions.

6. (0,0) and (1,10) Slope =

7. (0,0) and (4,2) Slope =

8. (−3,−9) and (0,0) Slope =

9. (1,1) and (5,5) Slope =

10. A line has a slope of 3. If it passes through (1,2) it must also pass through:

a. (0,−1)
b. (−2,1)
c. (0,3)
d. (4,3)

11. A line passes through two points: (a,b) and (c,d). The slope is:
[Be sure to check your answer using real points]

a. (b+d) / (a+c)
b. (d−b) / (c−a)
c. (c−a) / (d−b)
d. (a−c) / (b−d)
e. cannot be determined

12. What is the slope of the line that passes through (1,2) and (1,10)?

a. 8
b. −1/8
c. 1/8
d. −8
e. Undefined

Portfolio Chapter 10

An explanation of how coordinate systems work, in your own words, would be nice here. Create something that will help you remember this topic, because it will come back in Geometry, Algebra II, and Calculus.

The coordinate system we looked at gives us a way to describe any location on a flat surface. You may want to look up Polar Coordinates to see how that could be done a different way. And how about describing a location in a 3-dimensional space?

Chapter 11: Linear Equations

In the previous quiz, you saw that if there are two points on a line, like (a,b) and (c,d), you can find the slope by subtracting, like this: $\frac{d-b}{c-a}$, or, starting at the other point: $\frac{b-d}{a-c}$. By convention we use the letter m to represent the slope. Possibly this is related to the root word for mountain. Unsurprisingly, people often use the letter x for the x-coordinate of a point, and the letter y for the y-coordinate. That is easy if there is only one point, like (x,y), but more likely there are at least two points. Here we often see subscripts, like this:

(x_1,y_1) and (x_2,y_2)

That is a bit unfortunate because it tends to make things look a little more complicated

The official formula for calculating a slope is:

$m = \frac{y_2-y_1}{x_2-x_1}$. Look at this formula carefully. It should not really be necessary to memorize it, since you can re–create it at any time using your understanding of the subject.

And now, the moment you've all been waiting for ... or not. Getting back to the equation from the last chapter, x + y = 10, let's see if we can find more information about the two unknowns x and y. This may look familiar from when you first learned to add: 0 + 10 = 10, 1 + 9 = 10, 2 + 8 = 10, etc. Just by saying that x + y = 10, we have defined a specific relationship between these two unknowns. The interesting thing is that we can graph this relationship. We can graph x as the x–coordinate, and y as the y–coordinate in a Cartesian system. Go ahead and mark the points (0,10), (1,9), ... up to (10,0). When you were young, you would have thought that there are a very limited number of such points. Now that you know about negative numbers, you see that they go on forever. All of these points form a line that extends forever past (–1,11) and (11,–1). Notice that x and y are really no longer simple unknowns. They can assume many different values, so we call them **variables**. Although you can pick any value for one of these variables, once you do so the value of the other variable is already determined. By convention, we always pick the value for x first and then see what y turns out to be. In technical terms, that makes x the **independent variable**, and y the **dependent variable**. That is, y depends on x. Here the variables x and y have a relationship that graphs as a straight line, so we call x + y = 10 a **linear equation**.

Let's look at another linear equation: x + y = 0. Starting with x = 0, graph 5 points for this equation. When I tell students in person to graph an equation, they often seem confused. Graphing equations is really not that hard. You just need to some pick fairly small values of x so that both x and y can fit on a coordinate system drawn on paper. Once you have selected and graphed your points, you can probably see that it would be more convenient to write the equation as y = −x. This way we can put in a value for x, and out comes the value for y. For example, if I select −5 for x, y would be 5, so I graph the point (−5,5). When the equations are more complex, it really makes things easier to write y = ... Writing linear equations in this format is a natural thing to do. If people did this first, it probably very quickly led to another discovery, which we will recreate now.

The Slope Intercept Form

Take out a clean piece of graph paper, and graph the following equations:
y = x
y = 2x
y = 4x
y = 0.5x

Calculate the slope of all 4 lines. You should now notice that the value of the slope is already there for you, right in front of the x. In general terms, for any equation of the form y = ax, the slope is a. Since the number in front of the x always indicates the slope, and we like to use the letter m for the slope, we will write this general equation as y = mx.

Next, take another clean piece of graph paper, and graph:
y = x
y = x + 2
y = x + 6
y = x − 3

You can see what happens to the line if you add or subtract a number at the end of the equation. Notice where each line intersects the y–axis. The point of intersection with the y axis is also the point where $x = 0$. In the first equation, when $x = 0$, y is obviously also 0. The last equation returns a value for y of −3 when x is 0, which is why the line intersects the y–axis at −3. The y–intercept occurs when $x = 0$.

For the equation $y = mx + b$, the slope is m and the y–axis intercept point is b. [Sometimes you will see this same equation written as $y = ax + b$, in which case a is the slope.] $y = mx + b$ is called the **slope–intercept form**. Notice that you could also write this equation in a different way, maybe like this: $y - mx - b = 0$. It would still be a linear equation, but it would not be in slope–intercept form.

Finding the Equation of a Line

The slope intercept form is useful when you know the slope of a line, and you also have the location of one of the points on the line. For example, suppose that we know that a line has a slope of 5, and (6,11) is a point on the line. Use $y = mx + b$ and fill in the value of the slope, so you get $y = 5x + b$. But how can we find b? Well, the equation that we create has to work for any point on the line, so it should also work for the point (6,11). Because you have a point on the line, you can use these values for x and y *temporarily*, and calculate the only remaining unknown b. Use 11 as the y–value and 6 as the x–value in the equation:

$$11 = 5(6) + b$$

You should find that $b = -19$. Now take that value and plug it into the general equation for your line: $y = 5x - 19$. This equation works for the sample point (6,11), and also for every other point (x,y) on the line.

Horizontal and Vertical Lines

The slope of a horizontal line is zero. Suppose you have a horizontal line that crosses the y–axis at $y = 4$. For every value of x, y is 4. You can easily fill in the equation $y = mx + b$, since the slope m is zero: $y = 0x + b$ or $y = b$. In this case the description of the line would be $y = 4$. x is not in the equation, but that doesn’t matter because the y value is the same for every single value of x.

A vertical line has no slope. We can't use the standard equation y = mx + b to describe a vertical line, but it is easily done another way. Just as we can say that y = 4 to create a horizontal line, we can make a vertical line by saying that x = 4. Now x is 4 for every value of y, and the line is vertical. This vertical line crosses the x–axis at x = 4. For any vertical line, look where the line crosses (intersects) the x–axis and then use that value to create your equation.

X and Y Intercepts

A line intersects the y–axis at the point where x = 0. The intersect point is called the y–intercept. You can read the y–intercept directly from the equation y = mx + b by setting x equal to zero, which means that the y–intercept occurs at y = b.

Sometimes you may be asked where a line intersects the x–axis. This point is called the x–intercept. The x–intercept occurs when y = 0, so plug that into the equation of your line. For example, for the line y = x – 3, we can find the x–intercept by setting y equal to 0, like this: 0 = x – 3. Add 3 to both sides of this equation to find that x = 3. The line intersects the x–axis at x = 3.

Try this link for a faster look at slopes and intercepts:
http://www.shodor.org/interactivate/activities/slopeslider/

Look back at the graphs you made earlier, of the equations with different y–intercepts. All of the lines you drew are (hopefully) parallel, since they all have the same slope. Two lines that have the same slope will never intersect. Next we will consider lines that are perpendicular to each other, which is sort of the opposite of parallel. The x–axis is perpendicular to the y–axis in a Cartesian coordinate system. The two lines that make up a + sign are also perpendicular to each other. Let's draw some perpendicular lines now. Draw one line through the points (0,0) and (8,2). Next draw a line through (–2,8) and (0,0,). The slope of the first line is $\frac{2}{8}$ or $\frac{1}{4}$. The slope of the second line is $-\frac{2}{8}$ which is $-\frac{4}{1}$ or –4. These lines have "opposite" slopes. One line goes up as you look to the left (a positive slope), and the other goes up as you look to the right (a negative slope). The number of squares making up the rise and the run is reversed for the second line.

Draw the line representing the linear equation y = 2x, and the line for $y = -\frac{1}{2}x$. If you did that right, you will see that they are also perpendicular. After doing a few of these, you will realize that if you multiply the slopes of two perpendicular lines, you get −1. The slope of one line is the **negative reciprocal** of the other line. This fact is not particularly amazing as it is a natural result of how we have defined "slope", and "perpendicular". However, people who write tests on this subject really seem to like it. There is always at least one, and often two questions that test your knowledge of the fact that the slopes of perpendicular lines are each other's negative reciprocal. To save time, you should memorize this fact, as well as understand it.

This page explains regular reciprocals: http://www.mathsisfun.com/reciprocal.html. Remember to use the *negative* reciprocal for slopes of perpendicular lines so they will go in opposite directions.

The Point Slope Form

We already saw that if we know the slope, and a point on the line, we can easily find the equation of that line. For example, if the slope is 4 and a point on the line is (3,5), then we write:

$y = mx + b$

$y = 4x + b$ Temporarily insert the point in the equation so you can find b:

$5 = 4 \cdot 3 + b$

$5 = 12 + b$

$-7 = b$

The equation of the line is $y = 4x - 7$

There is another way to do this. Remember that we find the slope by using 2 points on a line:

$m = \frac{y_2 - y_1}{x_2 - x_1}$

Here we already know the slope, as well as one of the points. m = 4 and $(x_1,y_1) = (3,5)$. In this case we don't want to find a specific point (x_2,y_2). Instead, we are interested in the generic point (x,y) so we can write the general equation of the line. We will take (x_2,y_2) to be the generic point (x,y). Now the formula reads:

$4 = \frac{y-5}{x-3}$

Let's rearrange this by multiplying both sides by x – 3:

$4(x - 3) = y - 5$

$y - 5 = 4(x - 3)$

$y - 5 = 4x - 12$

$y = 4x - 7$

In general, you can find the equation of a line by rearranging the slope formula:

$m = \frac{y_2 - y_1}{x_2 - x_1}$ Multiply both sides by $x_2 - x_1$:

$m(x_2 - x_1) = y_2 - y_1$

$y_2 - y_1 = m(x_2 - x_1)$

Now use the generic point (x,y) instead of (x_2,y_2):

$\mathbf{y - y_1 = m(x - x_1)}$

This is called the **point slope form** of a linear equation, and yes you will need it in the future.

To use this form of a linear equation to solve problems, you again need the slope m, as well as a point on the line. Let's use the same example we used for the slope–intercept form. The slope was 5, and a point on the line was (6,11)

$y - y_1 = m(x - x_1)$

$y - y_1 = 5(x - x_1)$

Here we use (6,11) as the point (x_1,y_1):

$y - 11 = 5(x - 6)$

See if you can rearrange this equation to be the same as the slope–intercept equation $y = 5x - 19$ that we found in the previous section. Many students find the point slope form less confusing, because you can just stick the sample point into the equation and leave it there.

The point–slope form is actually always equivalent to the slope–intercept form. Let's look at why this is so. Back in the chapter "Equations with Two Unknowns", we learned to solve a problem by taking two equations and using substitution, addition or subtraction to get the answer. I know you really enjoyed that, so let's do it again.

$y = mx + b$ (The general equation of the line)

$y_1 = mx_1 + b$ (A point (x_1,y_1) inserted in the equation)

Although it is not possible to find some numerical solution for this system of two equations, an interesting thing happens when we try to solve it. It would be possible to subtract something from both sides of the first equation without really changing it. We could subtract y_1 from both sides, or, since y_1 is equal to $mx_1 + b$, we can subtract y_1 from one side and $mx_1 + b$ from the other side:

$$\begin{array}{rcl} y & = & mx + b \\ -\,y_1 & = & -\,(mx_1 + b) \\ \hline \end{array}$$

$y - y_1 = mx - mx_1 + b - b$

This eliminates b, and we can factor out m from the remaining terms on the right:

$y - y_1 = mx - mx_1$

$y - y_1 = m(x - x_1)$

Now we have changed the slope intercept form of a linear equation to the point–slope form. This should help you see that you can use either $y = mx + b$ or $y - y_1 = m(x - x_1)$ to find the equation of a line.

Direct Variation

When $b = 0$ in the equation $y = mx + b$, the equation becomes $y = mx$. The line of the graph goes through the origin. This is a special case of a linear equation. For this simpler case where $y = mx$, we can say that y varies directly with x. Because $y = mx$ is such a special case, people like to use k for the slope of the line instead of m:

$$y = kx$$

k is a common symbol used for constants, and it emphasizes that the ratio $\frac{y}{x}$ is constant in this special case. That is, if you take the equation $y = kx$ and divide both sides by x, you get:

$$\frac{y}{x} = k$$

This says that for any y and any x on the line, if you divide y by x the result is always the same. Don't just take my word for that! Try it out by creating your own equation and checking several points on the line.

So, for any two points on the line, $\frac{y_1}{x_1} = \frac{y_2}{x_2}$ (and that works the other way around too: $\frac{x_1}{y} = \frac{x_2}{y_2}$).

This works only for this special case. Again, create your own equation $y = mx + b$ and see if the ratio $\frac{y}{x}$ is actually not constant when b is not zero.

Problems involving direct variation are usually easy to solve. All you have to do is find the constant k, which involves a simple division: $\frac{y}{x} = k$. Once you have k, use it to find the value the problem is asking for.

Read more and get some practice here:

Direct Variation Word Problems (moomoomath.com)

Systems of Equations

In Chapter 9 we looked at solving equations with two unknown quantities. To solve such equations you need at least two pieces of information about those unknowns. Each piece of information can be written as an equation. Two or more equations about the same unknowns are called a system of equations.

Consider the following system of equations:

$y = 2x + 7$

$y = -5x$

If you use substitution, you can take the value −5x to substitute for y, so you get

$-5x = 2x + 7$

If you feel comfortable with your algebra tools, you should have no problem finding that $x = -1$. Since $y = -5x$, that means $y = 5$.

If you use subtraction instead, you get

$$\begin{aligned} y &= 2x + 7 \\ y &= -5x \\ \hline y - y &= 2x - -5x + 7 \end{aligned}$$

This means that $0 = 7x + 7$ and x is still −1. The answers are x = −1, and y = 5. Always check your answers to make sure that they work in both equations.

Graph these two equations carefully. The two lines will intersect at (−1,5). Now that's really amazing. Graphing can be used to solve a system of equations because the intersect point satisfies both equations. Graphing by hand is a bit slow, but graphing calculators and computer software can do this faster than you can use your algebra to find a solution. Free apps are available for download to your computer or smartphone, or for use online. This gives you another way to check your solutions to problems involving two equations.

Graphing Linear Inequalities

If you know how to graph linear equations, you can also learn how to graph linear inequalities. It really isn't that hard. Check it out here:

http://www.purplemath.com/modules/ineqgrph.htm

Make sure to check out the section on systems of linear inequalities at this site too. Note that you can check your work by selecting a random point in the shaded region and verifying that it satisfies the inequality.

Portfolio Chapter 11

Phew, this was a long chapter with a lot of stuff that you'll need later on in your studies. Below is a summary to help you create one or more portfolio pages. You should write or copy an example for each point.

Summary

The slope of a line is given by the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$.

$y = mx + b$ is the slope–intercept form of a linear equation. The slope is m and the y–intercept is b.

Point Slope Form: $y - y_1 = m(x - x_1)$

When you multiply the slopes of perpendicular lines, you get –1.

The point where two lines intersect is the solution to the system of linear equations that describes these lines.

Direct Variation: $y = kx$. Find k by dividing: $k = \frac{y}{x}$

For a linear inequality, shade above or below the line you get from graphing the equivalent equation.

Chapter 11 Quiz

1. Consider the line described by the equation y = −x + 1. Are the following points located on this line?
(−5,6)
(4,−5)

2. What is the slope of the line defined by y = −x + 1?

3. The line defined by y = −x + 1 intersects the y−axis at the point __________

4. The line defined by y = −x + 1 intersects the x−axis at the point __________

5. Find the equation of the line that has a slope of 2 and passes through the point (3,5).

6. Once you select two points to draw a line, you have completely defined this line. Therefore, it should be possible to come up with a linear equation to describe this line once you have your two points. Let's take two points, (2,11) and (4,21) and write an equation in the form y = mx + b that describes the resulting line.
First, calculate the value of m. m = ______. Now what you have left is an equation with 3 unknowns. How can we possibly solve this?? Remember that there are many related values of x and y that will belong in this equation. We can just use a pair that is already provided for us, like (2,11) or (4,21). When we fill in values for x and y, b will be the only unknown, so we can find what it is.
y =_____ x + _____ Graph this equation to check your work.

7. Draw a line through the points (1,–5) and (5,–13). Find the linear equation that describes this line.

y = ____ x + ____

8. Find the equation of the line that is perpendicular to the previous line, and also passes through the point (1,–5). You can use fractions or decimals in your answer.

y = ____ x + ____

9. Consider the lines described by the following two linear equations:

$y = 4x - 3$

$y = -3x + 11$

At what point do these lines intersect? Check your calculated answer by graphing.

10. Matthew wants to save up for a new gaming console. He has $80, and he plans to set aside $10 per week. What is the equation of the line he could graph to show his total savings over time?

11. Solve the following system of equations:

$y + x = 5$

$y = \frac{1}{3}x + 1$

12. Direct Variation: y varies directly with x, and (5,15) is a point on the line. What is the value of k? Write the equation for the graph.

Chapter 12: Functions

When we looked at linear equations, like $y = 3x + 2$, we saw that there are infinitely many pairs (x, y) for which the equation is true. Once you select a value for x, there is a specific value for y that goes with that. By picking some random small values of x, you can easily draw a graph of this equation, and you should find that it is a straight line with a slope of 3 and a y–intercept of 2. There are other equations involving y and x that don't graph as a straight line, such as $y = x^2$. Locate some graph paper and graph $y = x^2$. Make sure to use some negative values for x as well as positive ones. Next, graph $y = x^3$, again using both negative and positive values for x.

After you draw a few graphs like that, you may wonder if there is an easier way, and there is. There are many excellent free apps for your computer or smartphone that will do the job. The way to input equations with powers is to type y = x^2 or y = x^3. One program I like to use on my computer is MathGV. At the MathGV website, click on the download link on the left side of the page. To use this program, go to "File", "new 2d Cartesian graph", and then select "Graph" – "new 2D function". Enter the part that comes after y =, like x^2 or x^3. There are also a number of online graphing calculators so you don't have to download anything. Once you have the graphs for $y = x^2$ and $y = x^3$, check that you drew them correctly on paper. You need to be familiar with the appearance of these two important graphs.

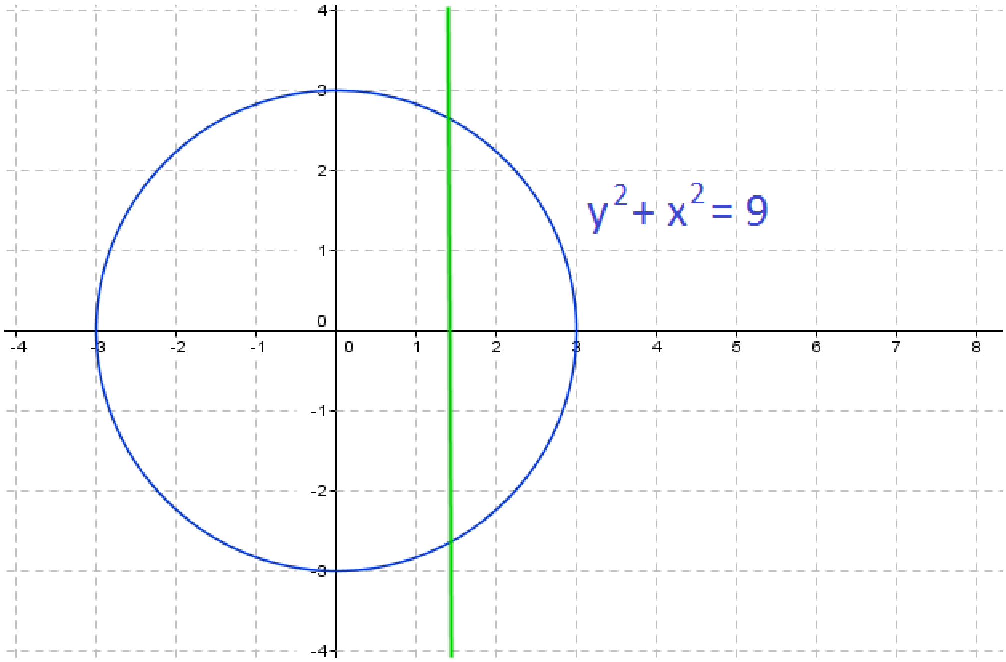
Most equations that we write using y and x give a single value for y when you put in a value for x. Such equations are called functions. The equation $y^2 = x$ also represents a relationship between y and x, but for any value of x except zero you get two different values for y. For example, if x is 9 then y will be 3 or −3, and if x is 2 then y is $\sqrt{2}$ or $-\sqrt{2}$. To see the problem with that, let's think of an x–y relation as a machine, like a vending machine. Regular vending machines require money, but numbers are not nearly as popular as snacks or soft drinks and you can hardly get people to take them for free. So, we'll ignore the money part and imagine some vending machines. Just push the buttons, and out comes what you want.

Fred is stocking the vending machines with soft drinks, and he is very experienced. He makes sure all of his machines supply cola, a lemon–lime soft drink, orange soda, and root beer. Each machine has four large buttons with the proper labels. Because the machine at

8th Avenue and 5th Street always runs out of cola, and some woman yells at him about it, Fred has replaced the root beer section of this machine with a second cola slot. Rita, who is addicted to cola, is very happy with this arrangement. She can push either button 1 or button 2 to get the soft drink she wants. One day, management decides that consumers want healthier choices. Fred receives instructions to replace root beer, which has not been selling well, with bottled water. He dutifully places water bottles in all the root beer slots, and changes the label on the buttons. However, when he gets to the machine at 8th Avenue and 5th Street, he isn't sure what to do. He doesn't want to be yelled at, but management has made it clear that all machines must contain bottled water. Finally he decides to just mix some water bottles in with the cola in the second slot. By the time Rita gets to the machine, slot 1 is empty. The second button still has a cola label, but a small hand–written note is taped below it. The note says "or bottled water." Now, how do you think Rita will feel about this new arrangement? Although it would be better for her to get water once in a while when she pushes the button, the randomness of the outcome is likely to cause her to be quite angry with poor Fred.

Mathematicians don't like surprises either, so they carefully define functions to ensure a unique output for every input. This does not mean that two different x's could not return the same y value, because that outcome is predictable just like having both button 1 and button 2 provide cola. The function $y = x^2$ gives a value of 9 when x is 3, and also a value of 9 when x is –3, and that's fine. The only thing that doesn't work for a function is the unpredictable result we would get with relations like $y^2 = x$, or $x^2 + y^2 = 10$. For $y^2 = x$, suppose x = 4, which means that y is either 2 or –2. That is a lot like getting either water or cola, with no way to control the outcome ahead of time.

When a relation is written out it is usually obvious whether it is a function or not. If all you have is a graph, you may want to apply the **vertical line test**. To do that, you simply draw, or pretend to draw, a vertical line across the graph. If any vertical line touches the graph in more than one spot the graph does not represent a function because there is more than one y–value for a particular x. In the picture below, a vertical line has been drawn across the graph of $y^2 + x^2 = 9$. This graph is a circle, and it does not represent a function because the line touches the graph in two places. To draw a circle using functions you need two of them: $y = \sqrt{9 - x^2}$, and $y = -\sqrt{9 - x^2}$.



Why are Functions Useful?

Suppose that on your birthday you and a group of 7 friends go to play mini golf at an adventure golf course that charges \$11.49 per person. To determine the total bill, the person at the cash register simply pushes two buttons and tells you that the total cost is \$91.92. There is no need to use a calculator because the register is computerized. All the employee has to do is enter the number of players and ask for the total. This is possible because we can communicate to the computer how the total cost is to be calculated. To do that, programmers must use functions. An appropriate function for this situation is y = 11.49x, where x is the number of players. Other letters may be used, so the function could look like T = 11.49p, where T is the total cost.

Mini golf courses may offer discounts, such as a 10% off coupon. A separate function can apply the discount. To calculate that discount you would probably take the total, \$91.92, and find 10% of that which is \$9.19. Then you would subtract \$9.19 from \$91.92 to get the new price. Think about how you would create a function that would take any total T and calculate a new discounted price D.

You would probably start by finding 10% of T, which would be written as .1T. That has to be subtracted from the original total T to get the discounted price D, so D = T – 0.1 T. If you look at that for a bit you might notice that you could just as well write D = 0.9 T. However, it would also be a good idea to allow for a flexible discount so you could use the first function and add an additional variable that represents the discount percentage.

Functions are useful for more than calculating totals. There are several methods that find the maximum or minimum value of a function, so businesses can use functions to maximize profits, build more efficient machines, or minimize packaging waste. Because functions are so important, college admission tests now include a fair number of questions that require you to know a lot about them.

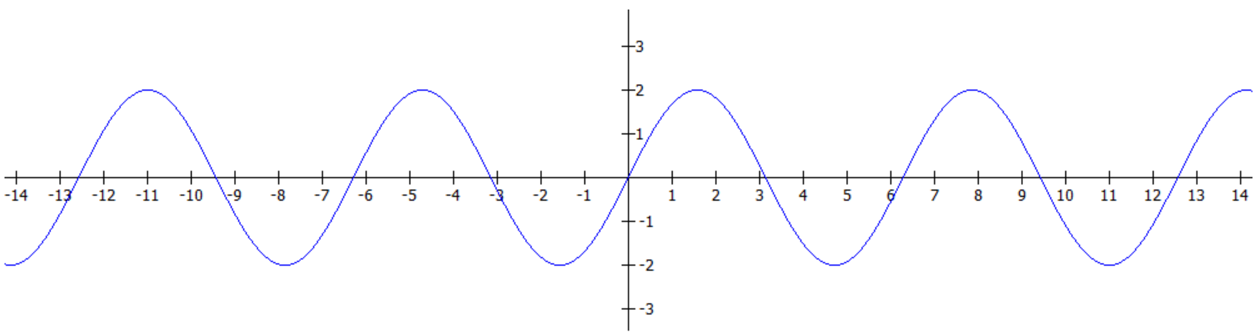
The Domain

The word domain means an area that is controlled by someone or something. The domain of a function refers to all of the input values that the function is able to use and produce an output for. When you first start learning about functions, your textbook may give you specific values to put into the function that you are using. The domain is often restricted to the given values only. Later on, you will be expected to find your own sample values for x to graph the function, and you will consider which numbers are possible values of x that you can actually use.

A function like $y = x^2$ can use any number x to create a value for y, so we say that the domain is "all real numbers". Real numbers include zero, regular positive and negative numbers, and irrational numbers like $\sqrt{2}$ and π (more about those later). Not all functions have an unlimited domain. For example, the function $y = \frac{1}{x}$ cannot use 0 as an x value because that would cause division by 0, so the function is not capable of producing an output when x is 0. The domain of this function is $x < 0$ and $x > 0$, or we can simply specify that $x \neq 0$. Another function that does not have an unlimited domain is $y = \sqrt{x}$. Negative numbers don't have real square roots, so you would specify the domain as $x \geq 0$. It may seem a little more complicated to find the domain for something like $y = \sqrt{x - 10}$, but all you really have to do is to make sure that the expression under the square root sign is greater than or equal to zero. Just use an inequality: $x - 10 \geq 0$. That solves as $x \geq 10$, which gives you the domain of the function.

The Range

The range of a function refers to the range of the output values that it can produce. Just think of it as "where the function can go." If you are given only certain values of x to use, the range is simply the collection of all the numbers you get for y. If no specific values of x are given, try to figure out the lowest and the highest possible values for y. For example, the function y = x^2 can take any x as input, but the output will never be negative. The range includes all numbers from zero to infinity. y = $\sqrt{x - 10}$ has a different domain than y = $\sqrt{x}$, but its range is the same. The smallest output produced by both these functions is 0, and there is no limit on the largest output. You could describe the range as $y \geq 0$. To find the range of a function from a graph, look for the smallest and the largest y value. For example, in the graph below the y values go from −2 to 2, and the range can be described as $-2 \leq y \leq 2$.

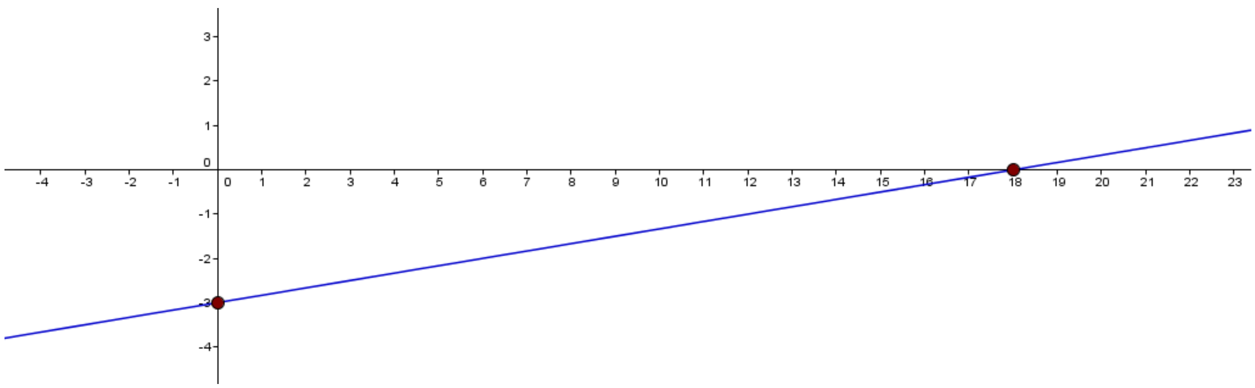


Even though this graph may look a bit odd, you can be sure that it is a function because it passes the vertical line test. There is only one y value for every x.

The Intercepts

You will often be asked to find the x and y intercepts for a particular function. That means you need to determine where the graph crosses the x-axis, and where it crosses the y-axis. The y-intercept is usually easiest to find. Somehow, the y intercept always occurs when x is zero (make sure you can see why). Anyway, all you have to do is let x be zero, and you will get the y-value of the y intercept. By an equally amazing coincidence, the x-intercept always occurs when y is zero. Just choose 0 for y, and determine the x-value.

Let's see if we can calculate the x and y intercepts for the function $y = \frac{1}{6}x - 3$. When the graph crosses the y-axis, x will be zero. Just let x be zero: $y = \frac{1}{6}(0) - 3$. Since $\frac{1}{6} \cdot 0$ is zero, y = -3. The function crosses the y-axis at y = -3, when x is zero. The intercept point is (0, -3). To find the x intercept, let y be zero: $0 = \frac{1}{6}x - 3$. This means that $3 = \frac{1}{6}x$. We can now multiply both sides by 6 (or divide by 1/6 which works out the same), to get 18 = x. The x intercept point is (18, 0). You can see the intercept points on the graph below:



Function Notation

So far, you would probably say that functions don't seem all that difficult to handle. Unfortunately, as people worked with functions regularly they needed different notation. First of all, if you are using two or three different functions you may want to name them to keep track of which one is which. Let's say we are using two functions, $y = x^2$ and $y = 3x$. We can call the first function f, and the second function g. Now, when x is 4, the output of the first function is y = 16, and the second function returns y = 12. It's not good that both these outputs are called y, so we could use subscripts and call them y_1 and y_2, or maybe y_f and y_g. However, there are some problems with that. Subscripts are small, make things look more complicated, and are a pain to type. Also, when you say that $y_f = 16$ and $y_g = 12$, that doesn't contain any information about which value of x you used to get those outputs. The notation that people eventually settled on is this: f(4) = 16 and g(4) = 12. This says that the output of the function f, when x is 4, is 16, and the output of the function g when x is 4 is 12. In words, it is read as "f of 4 is 16", and "g of 4 is 12." The functions themselves are written like this: $f(x) = x^2$, and $g(x) = 3x$. Please note that in this case the parentheses are simply used to contain the value of x. f(x) does **not** mean f times x!

Many students who have no trouble with the x–y format for functions find that problems suddenly seem much harder when presented in function notation. If that happens to you, just replace f(x) or g(x) with y to get back to a more familiar format. After a while you should get used to seeing f(x) and g(x) and they won't look so scary anymore.

Example 1: If $y = 3x + 10$, find y when x is 5.

Solution: Substitute 5 for x in the equation, so $y = 3(5) + 10$, which means $y = 25$.

Example 2: If $f(x) = 3x + 10$, find f(5).

Solution: You might notice that this is really exactly the same problem as the one in example 1. Substitute 5 for x in the equation, so $f(5) = 3(5) + 10$, which means $f(5) = 25$.

Example 3: If $f(x) = 3x + 10$, find f(4x).

Solution: Function notation allows us to create abstract problems like this, which is good for people who write math problems for a living because that gives them more options. For you, it is not necessarily bad, especially if you are good at using substitution. Here we will use the given value, 4x, to substitute for x in the equation. You will want to use parentheses: $f(x) = 3(x) + 10 \rightarrow f(4x) = 3(4x) + 10$. This means that $f(4x) = 12x + 10$.

Sequences

A sequence is an ordered progression of numbers. The simplest kind of sequence is an arithmetic sequence. Arithmetic sequences are linear patterns. When you graph them the points are all in a straight line. This happens because arithmetic sequences progress by adding the same constant amount each time. An arithmetic sequence looks like this:

3, 5, 7, 9, 11, …

If you look carefully, you can see that the next number in this sequence should be 13, because the numbers are increasing by 2. Here 2 is called the common difference. The letter d is often used to indicate this difference. For a sequence like 1, 4, 7, 10, 13, …, the common difference is 3. To find the common difference if it is not immediately obvious, take any term and subtract the previous term. Keep in mind that the difference d could be a negative number, in which case the numbers of the sequence keep getting smaller. The difference can also be a fraction or a decimal, as in: 9, 8.9, 8.8, 8.7, 8.6, ….

If you have to create your own sequence you should show at least five terms of it so that the pattern of your sequence is relatively clear to someone else. You may think you are showing an arithmetic sequence, but if you provide too few numbers someone else could come up with a different kind of pattern that also fits those same numbers.

Let’s take a closer look at the sample sequence 3, 5, 7, 9, 11, …. The first term is 3, the second term is 5, the third term is 7, the fourth term is 9, and so on. You could make a table:

Term	Value
1	3
2	5
3	7
4	9
5	11

Because each term has one specific value, we can actually consider a sequence to be a function. Although you can easily work with sequences without thinking of them as functions, educators want you to use function notation here so you can practice using it.

Term	Value
f(1)	3
f(2)	5
f(3)	7
f(4)	9
f(5)	11

Terms, of course, have to be numbered using whole number values rather than just any kind of number. In algebra we usually use x for variables, or for the input of functions, but for whole numbers it is customary to use n:

n	Term	Value
1	f(1)	3
2	f(2)	5
3	f(3)	7
4	f(4)	9
5	f(5)	11

So, when we talk about some random term in the sequence, we call it the n^{th} term. If n happens to be 5, then the n^{th} term is the 5^{th} term. When n = 100, the n^{th} term is the 100^{th} term. Sequences may be represented like this:

f(1), f(2), f(3), f(4), …, f(n), …

Sometimes we want to talk about the term that comes just before the n^{th} term. That would be the $(n-1)^{th}$ term, or f(n-1). The term just after the n^{th} term would be f(n+1), and so on:

f(1), f(2), f(3), f(4), …, f(n-1), f(n), f(n+1), f(n+2), …

To find the next term in an arithmetic sequence, take the last available term and add the common difference. Keep doing that. In computer science, you would accomplish this by creating a loop in your program that the computer keeps following over and over until some specified condition is met. This is called **recursion**. The recursive formula for finding each successive term just says in algebra what we just said in words: to find any arbitrary term f(n), take the previous term f(n-1) and add the difference d:

$$f(n) = f(n-1) + d$$

If you also know the first term, f(1), you can generate the sequence from the recursive formula:

f(1), f(1) + d, f(2) + d, f(3) + d, f(4) + d, f(5) + d, …

As an example, we will find a recursive formula for the arithmetic sequence
3, 5, 7, 9, 11, ….

You could also write this sequence as 3, 3 + 2, 5 + 2, 7 + 2, 9 + 2, etc. Each term is created by adding 2 to the previous term. The recursive formula is:

$f(n) = f(n-1) + 2$, and $f(1) = 3$.

This says: "Start at 3 and find each next term by adding 2 to the previous term."

Recursive formulas are easy for computers to use. They can find the 1000th term of a sequence in just a fraction of a second, but humans don't calculate that fast. If we want a specific term of a sequence it is better to use an explicit formula. In this case the word explicit means straightforward, or stated clearly, without all of the roundabout calculations required for the recursive method. An explicit formula that generates the sequence 3, 5, 7, 9, 11, …. would be $f(n) = 2n + 1$. Notice how easy it is to find the 5th term by using this formula: $f(5) = 2(5) + 1 = 11$. The 1000th term is $f(1000) = 2(1000) + 1 = 2001$.

To figure out what the explicit formula should be, I used the sequence 2, 4, 6, 8, 10,…. This sequence also adds 2 to each term, but it is simpler because it just starts at 2. This sequence is just $1 \cdot 2, 2 \cdot 2, 3 \cdot 2, \ldots, n \cdot 2, \ldots$. It can be generated by the formula 2n, which we can write in function notation as $f(n) = 2n$. Although there technically isn't a 0th term, if there was it would just be 0. For 3, 5, 7, 9, 11, …, the 0th term would be 1. That tells me that this sequence is just 1 bigger, for each term, than 2, 4, 6, 8, 10,…. I add 1 to my formula to get $f(n) = 2n + 1$. Just look back at Chapter 3, "Patterns" to see more examples of how to create the right formula to make arithmetic sequences.

Chapter 12 Quiz

1. On paper, create a graph of $y = x^3 + 3$. Is this a function? Why or why not?

2. Is $x^2 + y^2 = 4$ a function? Why or why not?

3. Is $x + y = 4$ a function? Why or why not?

4. Bobo the clown charges $98 to entertain at a birthday party, plus an additional $2.99 per child to provide balloon animals to each guest. Write a function that describes his total fee for a party with c children.

5. What is the domain of $y = \sqrt{x+4}$? What is the range?

6. Write the function $y = x^2 - 12$ in function notation. What is y when $x = 1$?

7. For the function $f(x) = 5x + 2$, find $f(3)$.

Portfolio Chapter 12

If you are into computer programming, you may want to write some interesting code here and show the output. Otherwise, there are many sequences you can create and write formulas for.

Think of a real-world example where a function would be useful. What is the domain of your function?

Chapter 13: Exponents

This lesson contains a set of rules for working with exponents. It is not necessary to memorize these rules. Using your basic algebra tools, you can re–create them at any time. Or if you feel lazy you can always look them up on the internet.

xxxxxxx · xxxxx = xxxxxxxxxxxx. This may be a true statement, but it is a terrible way to write it. It is much better to use exponents [also called powers]. Recall that $x \cdot x = x^2$ or x to the 2nd power [The · is optional and is only included for clarity]. In the same way, x^3 means $x \cdot x \cdot x$ and x^5 is a much quicker way to write xxxxx. It is also easier to read. Note that x^1 just means x, and we rarely write it that way.

Exponents are the second item in "**P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally". This means that they have a very high priority in the order of mathematical operations. An exponent on something generally belongs exclusively to that one thing, unless there are parentheses present. So, $3x^2$ is just $3x^2$, because the exponent applies only to the x. However, once you place some parentheses they get priority, and the exponent applies to the entire thing inside those parentheses. $(3x)^2$ means $3x \cdot 3x$ which is $9x^2$.

Because exponents are used quite commonly in algebra, we need to be able to manipulate them quickly and efficiently.

First we will do some multiplication involving exponents, which is the easiest part.
$x^3 \cdot x^4 = ?$ This really means $x \cdot x \cdot x$ times $x \cdot x \cdot x \cdot x$, which equals xxxxxxx. Now you have 7 x's in a row, which is x^7. Notice that the exponents simply add up. $x^3 \cdot x^4 = x^7$. This also means that $x^5 \cdot x^5 = x^{10}$.

Once mathematicians figure something like this out, they can't resist creating a formula, like $x^a \cdot x^b = x^{a+b}$. Make sure you always refuse to be intimidated by such formulas that simply restate what was already explained. Say something like, "Yeah dude, you just said that, duhh!" Then look it over carefully, just to .. uh .. make sure the mathematician hasn't made a mistake .

Division with exponents is not much harder. We already know that $\frac{5x}{x} = 5$, because if you multiply by x and then divide by x, you have effectively done nothing. $\frac{5xx}{xx}$ is also 5. Just cross off one x above the line and one below the line, until all of the useless x's are gone. $\frac{xxx}{xx} = x$. We can also write this as $\frac{x^3}{x^2} = x^1$, which is just x. Another example: $\frac{x^5}{x^3} = x^2$ because we can remove 3x's above the division line and 3 x's below the division line. Because of how this works, we can quickly divide by subtracting the exponents. $\frac{x^7}{x^4} = x^{7-4} = x^3$. Write this out completely and make sure that you believe it is correct. The general formula for this is of course [ ]: $\frac{x^a}{x^b} = x^{a-b}$.

Notice that using this method, $\frac{x^1}{x^1}$ is x^{1-1} which is x^0 [note that x here cannot be 0, because that would involve dividing by 0]. Conveniently, x^0 is defined as 1, which is lucky since $\frac{x}{x} = 1$ [Any number divided by itself, like $\frac{3}{3}$ or $\frac{5}{5}$, is 1.] That means it makes sense to write $\frac{x^1}{x^1} = x^0$. We can use words like "conveniently" and "lucky" if we believe that mathematics is something we create. Actually it is probably something we discover, and our definitions end up the way they do because of what is already there, as part of the underlying structure of our universe.

Another thing that happens when you start subtracting exponents is that you end up with negative exponents. Consider $\frac{x^2}{x^5}$. You can write that as

$$\frac{xx}{xxxxx} = \frac{xx \cdot 1}{xxxxx}$$

Once you have removed two sets of x's that are not doing anything, you end up with 3x's below the line, or $\frac{1}{xxx}$. [Notice that there is still a 1 left once you remove the x's on top, not nothing!] If you just subtract the exponents, as we learned in the last paragraph, that would leave you with x^{-3}. This tells you that

$$x^{-3} \text{ means } \frac{1}{x^3}.$$

That's just the way things work out. So x^{-a} is $\frac{1}{x^a}$.

Because an expression like x^{-4} is really a fraction, you should treat it like one.

$\frac{x^{-4}}{5}$ means x^{-4} divided by 5: $\frac{1}{x^4} \div 5 = \frac{1}{x^4} \div \frac{5}{1} = \frac{1}{x^4} \cdot \frac{1}{5} = \frac{1}{5x^4}$

$\frac{6}{x^{-2}}$ means $6 \div \frac{1}{x^2} = \frac{6}{1} \cdot \frac{x^2}{1} = \frac{6x^2}{1}$

Multiplying with exponents causes us to have to add the exponents. You can also [trust mathematicians to think of this] put an exponent on something that already has an exponent. This is also called raising a power to a power, and it looks like $(x^2)^3$. Thinking carefully about what this means, we rewrite it as $x^2 \cdot x^2 \cdot x^2$ = xxxxxx or x^6. Basically, we end up with $x^{2 \cdot 3} = x^6$. So, when you raise a power to another power, you multiply the exponents. $(x^2)^3 = x^6$.

Be careful if there are multiple things inside the parentheses. $3x^2$ is just $3 \cdot x^2$, or $3 \cdot x \cdot x$. Order of operations rules say that the exponent must be applied first, followed by the multiplication by 3. However, $(3x)^2$ means $3x \cdot 3x$, which is $9x^2$. Everything inside the parentheses must be squared. In general:

$(xy)^a = x^a y^a$

To learn about fractional exponents visit the chapter on roots.

Scientific Notation

Visit this website for a thorough explanation of scientific notation:

http://ieer.org/resource/classroom/scientific–notation

Scientific notation also helps by getting rid of those potentially confusing zeroes at the beginning of numbers. If you write 0.03880 in scientific notation, it looks like this:

$0.03880 = 3.880 \times \frac{1}{100}$

Because $\frac{1}{100}$ can be written as 10^{-2}, the actual notation used is 3.880×10^{-2}. Now it is clear that 0.03880 has 4 significant figures. The zeros at the beginning of the number are not involved in showing how precise the measurement was.

Summary

$x^a \cdot x^b = x^{a+b}$

$\frac{x^a}{x^b} = x^{a-b}$

x^{-a} is $\frac{1}{x^a}$

$x^0 = 1$

$(x^a)^b = x^{ab}$

$(xy)^a = x^a y^a$

Chapter 13 Quiz

1. $4^2 + 2^3 + 5^0 =$

2. $-1 \cdot 3^2 =$

3. $-3^2 =$

4. $(-3)^2 =$

5. $x^2 \cdot x^3 =$

6. $x^3y \cdot xy^8 =$

7. $\frac{10^6}{10^4} =$

8. $\frac{x^{16}}{x^4} =$

9. $\frac{25x^3}{5x} =$

10. $\frac{x^{10}y^6}{x^4y^{12}} =$

11. $\left(\frac{1}{2}\right)^2 =$

12. $\left(\frac{4}{5}\right)^2 =$

13. $\left(\frac{x}{y}\right)^2 =$

14. $(3 \cdot 4)^2 =$

15. $(2b)^2 =$

16. $(x^3)^2 =$

17. $10^{-3} =$

18. $(x^2y^3)^4 =$

19. $\frac{x^{-10}}{x^5} =$

20. $\frac{4}{5^{-2}} =$

Portfolio Chapters 13 & 14

Exponents and their counterpart, roots, are often a source of confusion for students. As you continue to higher grades and perhaps to calculus, you'll need to be very comfortable with this stuff. Write down the rules and why they work, so you'll have something to look back to later on.

Chapter 14: Roots

One reason for studying roots is that a few square root signs can make your algebra formulas look much more impressive. Just look at Einstein's famous equation $E = mc^2$. [By the way, never let anyone shove a formula or equation in front of you without fully explaining what the letters mean. Here E stands for energy expressed in $\text{kg}\cdot\text{meters}^2/\text{sec}^2$, also called joules, m is the mass in kg, and c is the speed of light.] Using your algebra tools, you can rearrange that into $c^2 = \frac{E}{m}$. Taking the square roots on both sides, you get $c = \sqrt{\frac{E}{m}}$. Since the average person doesn't know this, you instantly appear smarter by being able to write this equation.

Roots are a natural and very ancient concept. One natural root is the side of a square. Take a square with sides of 5 inches. The area of the square is 25 square inches. The side is the square root of the area, since $5 = \sqrt{25}$. Because of this relationship, it was a natural thing for people to develop the idea of square roots, maybe while they were designing square buildings, or selling square plots of land.

By definition, the square root of a number is something that can be multiplied by itself to give that number: $\sqrt{9} \cdot \sqrt{9} = 9$. Because $3 \cdot 3 = 9$, the square root of 9 must be 3. In general, $\sqrt{x} \cdot \sqrt{x} = x$.

The Latin word for root is 'radix', so expressions or equations containing roots are called radical expressions or equations. Notice that the fact that the word radical also means something else does not appear to concern mathematicians at all. A statement like "today we'll be learning how to solve radical equations" can sound scary, but remember that roots are meant to be impressive, not scary. They are actually rather easy to work with.

Most of the roots you'll be encountering are square roots. Recall from Chapter 2 that we do not have any real number that we can multiply by itself to get a negative number. A positive number times a positive number gives a positive answer, and a negative number times a negative number also gives a positive answer. To get the square root of a negative number we will need imaginary numbers that you will learn about in Algebra 2. In this chapter we will only consider square roots of non–negative numbers.

Irrational Numbers and Proof by Contradiction

$\sqrt{16}$ = 4, $\sqrt{9}$ = 3, $\sqrt{4}$ = 2, and $\sqrt{1}$ = 1. Not all numbers have "nice" square roots, so that leaves a lot of interesting numbers in between. One of those numbers is $\sqrt{2}$. The actual value of $\sqrt{2}$ has to be somewhere between 1 and 2. 1.5 · 1.5 = 2.25, which is too big. 1.4 · 1.4 = 1.96, which is just a bit too small. If you keep working at this, you will find that you can come close to the right value, but you never actually get there. It is not possible to write $\sqrt{2}$ as a decimal value, like 14/10 or 141/100 or 1414/1000 etc. In fact, after struggling with the problem for a long time, people realized that it is not possible to write $\sqrt{2}$ in any form of one whole number, or integer, divided by another whole number. This seems odd when we consider that most numbers can be written as a ratio, or division, of two numbers. For example, 2 = $\frac{2}{1}$, and 0.5 = $\frac{5}{10}$ = 1/2. A number such as $\sqrt{2}$ and many other square roots are called irrational numbers to indicate that they can't be written as a ratio of two numbers. In fact, square roots of whole numbers that are not perfect squares like 9 or 25, are always irrational. π is another example of an irrational number. Because of the other meaning of irrational, it would be better to pronounce this word as "irraational", or change it to non–rational, but that just hasn't happened. We are stuck using the term "irrational numbers". Ancient records prove that people were aware of irrational numbers as early as 800 to 500 B.C.

Some interesting information: Hippasus of Metapontum, born circa 500 B.C. in Magna Graecia, was a Greek philosopher. He is the disciple of Pythagoras who is believed to have discovered existence of irrational numbers. He is also credited with figuring out that the square root of 2 is irrational. This discovery was made at sea and rumor has it that he was thrown overboard by fellow Pythagoreans, who did not want to believe that this beautiful, fundamental number could fail to be a ratio of integers. Well, maybe irrational is the right term after all...

When people failed to find two numbers a and b such that $\frac{a}{b} = \sqrt{2}$, they had to prove that it is not possible to do so. Some clever person devised a proof that no one could argue with. This proof makes use of the fact that if a^2 is even, a has to be even. Try that out with a few real numbers, and you'll see that it can't be any other way.

This proof is also a proof by contradiction. Such proofs may seem a bit confusing because you start by making some assumption and then showing that your assumption leads to a contradiction so that it was incorrect in the first place. Here we are going to assume that the square root of 2 is in fact rational so that it can be written as the ratio $\frac{a}{b}$. That assumption actually turns out to be wrong:

Assume that there are two numbers a and b such that $\sqrt{2} = \frac{a}{b}$ and $\frac{a}{b}$ is a fraction that is written in the lowest terms. There is no number that you could divide both a and b by to create a simpler fraction.

Now we can take the equation $\sqrt{2} = \frac{a}{b}$ and square both sides (raise both sides to the second power).

$\sqrt{2} = \frac{a}{b}$ becomes $\sqrt{2}^2 = \left(\frac{a}{b}\right)^2$

Since $\sqrt{2} \cdot \sqrt{2} = 2$, we get $2 = \frac{a^2}{b^2}$ which can be rearranged into $a^2 = 2b^2$.

This means that a^2 can be divided by 2, so it is an even number. If a^2 is an even number, then a must be even.

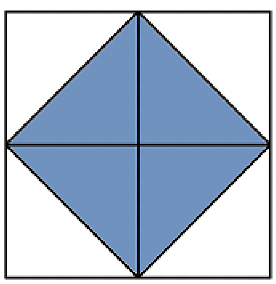
Now if a is an even number, there must be some number n such that $a = 2n$ [this is true for all even numbers]. Let's take the equation $a^2 = 2b^2$ from a bit earlier, and substitute 2n for a:

$(2n)^2 = 2b^2$.

Notice that $(2n)^2$ means $2n \cdot 2n$, which is $4n^2$. Therefore $4n^2 = 2b^2$ or $2n^2 = b^2$. Again, this means that b^2 has to be an even number, so b is even. But oops, a and b can't both be even numbers, because we said earlier that the fraction $\frac{a}{b}$ was in lowest terms; it couldn't be simplified further. [We should not be able to divide both a and b by 2.] This is a contradiction, so $\sqrt{2}$ cannot be $\frac{a}{b}$.

It is not abnormal to feel dizzy after reading this proof. Try reading this paragraph 9 more times. The dizzy feeling should go away, either before or after you throw up.

Although there is definitely something different about the number $\sqrt{2}$, that does not mean that it is not a real number. The square root of 2 is the side of a square with an area of 2 units. It is surprisingly easy to construct such a square. Start with a simple square that has sides of length 2, so that its area is 4. Divide your square up into 4 equal squares. Now divide each of those smaller squares in half, as shown in the picture below:



Since your original square had an area of 4, the shaded square must have an area of exactly half that, which is 2. The sides of the shaded square are $\sqrt{2}$ units long. **Irrational numbers are real numbers that have a place on the number line. Real numbers are either rational or irrational – there is no other kind of real number.**

Another irrational number is π, which is also written as pi, and pronounced as "pie", and again this is a real number. Because π is not rational we can't write it as a decimal number. The best we can do is an approximation: π = 3.1415926535… where the numbers go on forever and there is no particular pattern. The definition of pi is that it is the number that you multiply the diameter of a circle by to get the circumference. So, if the diameter of a circle is 5 inches, then the circumference is 5π inches or approximately 17.7 inches.

All numbers that don't have a nice square root have an irrational number as a square root. A number like $\sqrt{7}$ is irrational, and if there is an opportunity to do so we like to multiply it by itself to get a regular number: $\sqrt{7} \cdot \sqrt{7} = 7$.

When you add or subtract two rational numbers, you will always get a rational number. You can see that by writing the first number as $\frac{a}{b}$ and the second number as $\frac{c}{d}$. If you do a subtraction, you get $\frac{a}{b} - \frac{c}{d}$. Creating common denominators gets you to $\frac{ad}{bd} - \frac{bc}{bd}$, which is $\frac{ad-bc}{bd}$, a rational number. You can use the fact that a rational number minus a rational number is a rational number to prove that the sum of an irrational number and a rational number is always irrational. To do this, we use a proof by contradiction:

Suppose that an irrational number r plus a rational number $\frac{a}{b}$ is a rational number $\frac{c}{d}$. Then $r + \frac{a}{b} = \frac{c}{d}$, so we can say that $r = \frac{c}{d} - \frac{a}{b}$. But no, that won't work because the difference of two rational numbers is always a rational number, not an irrational number. Our assumption was wrong, so the sum of an irrational number and a rational number must be irrational instead of rational.

Negative Square Roots

Do not forget that numbers have a negative square root as well as a positive one. As we saw in Chapter 2, when we say $\sqrt{9}$, by convention we mean the positive square root of 9 which is 3. To indicate the negative square root we write $-\sqrt{9}$, which is −3. −3 · −3 is also 9, and −3 has just as much right as 3 to call itself a square root of 9. To make a contribution to the society for the Advancement of Negative Square Roots, ANSR, click here: http://www.ansr.org/

Clever students, such as those who did NOT click on the previous link, will realize that when we changed $E = mc^2$ to $c = \sqrt{\frac{E}{m}}$ we did not mention the negative square root $c = -\sqrt{\frac{E}{m}}$. This is because the speed of light is known to be a positive quantity, at least in this universe...

Most of the time there is no such restriction on a variable in algebra. To solve an equation like $x^2 = 9$, take the square roots on both sides: $\pm\sqrt{x^2} = \pm\sqrt{9}$. ± means plus or minus, and it indicates that the root can be either positive or negative. Now we get $\pm x = \pm 3$. Hmm, if both things are positive the answer would be 3. If they are both negative, $-x = -3$. Divide both sides by −1 to get $x = 3$, again. If the left side is positive and the right side negative, then $x = -3$. If the left side is negative and the right side is positive, then $-x = 3$. When you divide both sides by − 1 you get $x = -3$. Notice that there are actually only two outcomes here: $x = 3$ or $x = -3$. Because people don't like writing extra things the commonly used shortcut is to write $\sqrt{x^2} = \pm\sqrt{9}$ instead of $\pm\sqrt{x^2} = \pm\sqrt{9}$. After all, it's easier and doesn't affect the final result. Unfortunately however, students are usually left with the impression that $\sqrt{x^2} = x$. The problem with that is that it isn't always true, because if $x = -3$, then x^2 is 9, and $\sqrt{9} = 3$, not −3! For this reason, we write $\sqrt{x^2} = |x|$, so that there won't be a problem if x happens to be negative. $\sqrt{x^2}$ means the positive square root of x^2, and using the absolute value sign ensures that the answer will be a positive number.

Sometimes we forget that negative square roots exist, and we are surprised when they sneak into our equations. Consider the following equation:

$\sqrt{x^2} = -5$

The square root sign indicates the positive square root, which could never be a negative number like −3. However, suppose we don't notice this and attempt to solve the equation by squaring both sides:

$\left(\sqrt{x^2}\right)^2 = 25$

$x^2 = 25$

$x = 5$ or $x = -5$

When you put the answers back into the original equation, you can see that they are both incorrect. If you are solving a somewhat more complicated equation that contains a square root, you must solve it by squaring. In the process you may not notice the negative square root going along for the ride. Often you'll end up with two answers, one of which does not fit into the original equation. This is called an **extraneous solution**. You should always check your answers anyway by plugging them back into the original equation, so if you are used to checking your work this should not be a problem for you.

Sometimes there is more than one thing underneath a square root sign. Consider $\sqrt{ab}$. Extracting a square root from ab is a bit tricky, so watch me do it once before you try it yourself. When I say watch, we don't actually have a video so you'll have to imagine it like this:

We are standing in a clearing where I have just picked up the specimen ab. All around us the forest echoes with the sounds of unfamiliar creatures. Suddenly something rustles in the grass near our feet. We have to step aside quickly as the small half of an inequality slithers by, hungrily searching for prey larger than itself. Now, just look closely at ab, and you'll see that it actually has little roots. Here is a close–up: $\sqrt{a}\sqrt{a}\sqrt{b}\sqrt{b}$. Holding the specimen in my left hand I carefully rearrange the roots with my right, and separate them like this: $\sqrt{a}\sqrt{b} \cdot \sqrt{a}\sqrt{b}$. Now I grab one side, pull gently but firmly...and here is the square root of ab: $\sqrt{a}\sqrt{b}$. Not that hard, is it? The actual equation is $\sqrt{ab} = \sqrt{a}\sqrt{b}$, where a and b are either zero or positive numbers.

You can easily check that $\sqrt{a}\sqrt{b}$ is really the square root of ab; just multiply $\sqrt{a}\sqrt{b} \cdot \sqrt{a}\sqrt{b}$: $\sqrt{a} \cdot \sqrt{b} \cdot \sqrt{a} \cdot \sqrt{b} = \sqrt{a} \cdot \sqrt{a} \cdot \sqrt{b} \cdot \sqrt{b} = a \cdot b = ab$.

Try it out with some suitable numbers like $\sqrt{9 \cdot 16} = \sqrt{9} \cdot \sqrt{16}$ and convince yourself that it really works.

Using this same principle, you can simplify an expression like $\sqrt{50}$. It goes like this: $\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25} \cdot \sqrt{2} = 5\sqrt{2}$.

What if the question is a little harder, like: simplify $\sqrt{208}$? While some people may see almost instantly that 208 is 16 · 13, I'm just not such a wiz with numbers so I have to figure it out the hard way:

208 =

2 x 104. 104 =

2 x 52. 52=

2 x 26. 26=

2 x 13

This shows that 208 = 2 x 2 x 2 x 2 x 13. There are two squares here, 2^2 and 2^2, so we can change it to 2^2 x 2^2 x 13.

Now it is easy to take the square root: $\sqrt{208} = \sqrt{2^2 \cdot 2^2 \cdot 13}$ or $\sqrt{2^2} \cdot \sqrt{2^2} \cdot \sqrt{13}$ which is $4\sqrt{13}$.

Taking the square root of $\frac{a}{b}$ is not much harder than taking the square root of ab. The close–up looks like this: $\frac{\sqrt{a}\sqrt{a}}{\sqrt{b}\sqrt{b}}$, which can be separated into $\frac{\sqrt{a}}{\sqrt{b}} \cdot \frac{\sqrt{a}}{\sqrt{b}}$ [just multiply these two fractions to see that we haven't changed anything.] Then pull out the square root: $\frac{\sqrt{a}}{\sqrt{b}}$.

This means that $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ (neither a nor b are negative). Checking our work, we find that since $\frac{\sqrt{a}}{\sqrt{b}} \cdot \frac{\sqrt{a}}{\sqrt{b}} = \frac{a}{b}$ it makes sense that $\frac{\sqrt{a}}{\sqrt{b}}$ would be the (positive) square root of $\frac{a}{b}$.

While it may look fine to us to write something like $\frac{\sqrt{5}}{\sqrt{3}}$, math teachers get very upset if you leave a root in the denominator of a fraction. For this reason you should always remove such a root as soon as you notice it. Fortunately this can be done easily by multiplying both the top and the bottom of the fraction by the offending square root, as follows: $\frac{\sqrt{5}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$, =

$\frac{\sqrt{5}\cdot\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}}$. Since $\sqrt{3}\cdot\sqrt{3}$ is just 3, this can be written as $\frac{\sqrt{5}\,\sqrt{3}}{3}$. Remembering that $\sqrt{a}\sqrt{b} = \sqrt{ab}$, we simplify that to $\frac{\sqrt{5\cdot 3}}{3}$. Now our answer is $\frac{\sqrt{15}}{3}$.

Do not try to get around having a square root in the denominator by writing $\frac{\sqrt{5}}{\sqrt{3}}$ as $\sqrt{\frac{5}{3}}$. Leaving a fraction under a square root sign is also not acceptable to math teachers.

There are no tricks to rearrange $\sqrt{a+b}$ or $\sqrt{a-b}$. It is **NOT** true that = $\sqrt{a+b}$ = $\sqrt{a}$ + $\sqrt{b}$. We can easily see that since $\sqrt{9+16}$ is not equal to $\sqrt{9}$ + $\sqrt{16}$.

Do remember that regular math continues under the square root sign. This may allow you to factor something out of an expression and then move it outside the square root sign, like this: $\sqrt{4x+100}$ = $\sqrt{4(x+25)}$. Now you have two things under the square root that are multiplied by each other, so you can write $\sqrt{4}\cdot\sqrt{x+25}$, which is 2$\sqrt{x+25}$.

Cube roots are real things too, as they represent the side of a cube. $\sqrt[3]{27}$ = 3, since 3 · 3 · 3 = 27, and $\sqrt[3]{8}$ = 2.

Pulling out a cube root can be done using the same technique we used for square roots. Let's take the cube root of ab. Both a and b have third power roots; just look closely and blink, and there they are: $\sqrt[3]{a}\sqrt[3]{a}\sqrt[3]{a}\sqrt[3]{b}\ \sqrt[3]{b}\sqrt[3]{b}$. Rearrange like this: $\sqrt[3]{a}\sqrt[3]{b}$ · $\sqrt[3]{a}\sqrt[3]{b}$ ·$\sqrt[3]{a}\sqrt[3]{b}$. Now take out one of the three parts to get $\sqrt[3]{a}\sqrt[3]{b}$.

While cube roots work the same as square roots, there is an interesting difference. Numbers do not have both a positive and a negative cube root. A positive number has only a positive cube root, and a negative number only has a negative cube root. $\sqrt[3]{8}$ = 2 and $\sqrt[3]{-8}$ = –2.

Roots and Fractional Exponents

What about exponents that are fractions? Do they exist? Let's consider $x^{\frac{1}{2}}$. We know that when we multiply things with exponents, we can just add the exponents. So,

$x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1$.

This means that $x^{1/2}$ has to represent a number that, when multiplied by itself, is x. The only candidate for this is $\sqrt{x}$, since $\sqrt{x} \cdot \sqrt{x} = x$. There is no other choice than to conclude that

$x^{\frac{1}{2}} = \sqrt{x}$.

So what is $x^{\frac{1}{3}}$? There would have to be a number such that $x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} = x$. This is the number we call the cube root of x, or $\sqrt[3]{x}$. By now you can probably guess that $x^{\frac{1}{4}}$ is the fourth root of x, $\sqrt[4]{x}$. Creating a general formula: $x^{\frac{1}{n}} = \sqrt[n]{x}$ or the n^th root of x.

At some point you are going to see some very scary looking fractional exponents, like $x^{\frac{2}{3}}$. Do not despair! Just use your knowledge of fractions to see how such an exponent could have been created. We know that when we raise a power to a power, the two numbers are multiplied. That means that $x^{\frac{2}{3}}$ could have been created in two ways: $(x^2)^{\frac{1}{3}}$ or $\left(x^{\frac{1}{3}}\right)^2$.

Therefore, $x^{\frac{2}{3}} = \sqrt[3]{x^2} = \left(\sqrt[3]{x}\right)^2$. Both these forms mean the same thing. Because it is more trouble to use parentheses, you'll usually see $x^{\frac{2}{3}}$ written as $\sqrt[3]{x^2}$. Notice that it is the denominator of the fractional exponent that determines what kind of root is involved.

$x^{\frac{3}{5}}$ means $\sqrt[5]{x^3}$, and

$x^{\frac{a}{n}} = \sqrt[n]{x^a}$

Just remember that scary-looking exponents follow the same rules as regular exponents! If you have a complex problem like: simplify $\dfrac{x^{-\frac{1}{4}}}{x^{\frac{1}{4}}}$, go ahead and apply the regular rule for division which means subtracting exponents: $-\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$. Our answer is $x^{-\frac{1}{2}}$ which means $\frac{1}{x^{\frac{1}{2}}}$ or $\frac{1}{\sqrt{x}}$.

Eek, There Is a Root in My Denominator!

For some odd reason many people get upset if there are roots in the denominator of a fraction. This video shows how to take care of the problem:

http://www.youtube.com/watch?v=Thxzb35HQlw

Chapter 14 Quiz

1. The square roots of 25 are

a. 5 is the only square root of 25

b. 25 has no square root

c. 5 and −5

d. 1 and 25

2. $\sqrt{\frac{25}{9}} =$

3. $\sqrt{\frac{1}{4}} =$

4. $\sqrt[3]{27} =$

5. $\sqrt[3]{-125} =$

6. $\sqrt{4 \cdot 25} =$

7. Simplify: $\sqrt{8} =$

8. Simplify: $\sqrt{75} =$

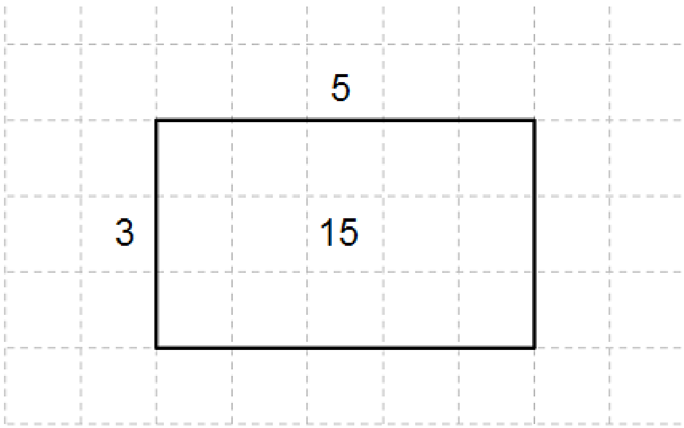
Chapter 15: More Multiplication

Note: The concepts covered in this lesson make nice portfolio pages. Label your drawings carefully and write the explanations and formulas on the same page. This will also help you remember things better.

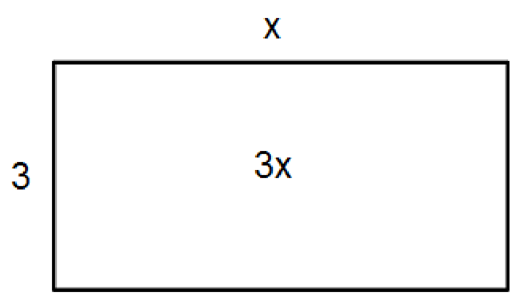
Suppose we have a problem with a square root, like $\sqrt{x} = 3 + a$. To get rid of the square root sign, we have to square both sides. We already know that $\sqrt{x}^2$ is x, but how do we square 3 + a ? Is it the same as $3^2 + a^2$? It should not be too difficult to check this out. Using a value of 4 for a, we get $(3 + 4)^2 = 3^2 + 4^2$, or $7^2 = 9 + 16$? That just doesn't look right.

Here you should remember that Algebra is not just a series of abstract formulas. It is a real thing that you can actually see and touch, and it was discovered by real people trying to solve real problems. The ancient Babylonians lived in Mesopotamia, between the Euphrates and Tigris rivers, around 1900 BC. They were very bright people. Not only did they know how to draw squares; they were also able to solve problems like $(a + b)^2 =$ ___. How was that possible so long ago? Well, as the old saying goes, necessity is the mother of invention. The Babylonians had merchandise to buy and sell, workers to pay, and taxes to calculate. Yet they had a number system that was not based on 10 like ours, but instead on the number 60. Part of their system survives today as 60 minutes in an hour and 60 seconds in a minute. Anyway, probably because of their complex number system [and a lack of calculators] they did not find it easy to multiply two numbers. In our number system, we can easily create times tables, and the first 10 times tables are quite sufficient to handle basic multiplication. Multiplying more complex numbers on paper is really a trick that makes use of place values. However, if you are using a base 60 number system you would need to memorize the first 60 times tables, and they would each have 60 entries!

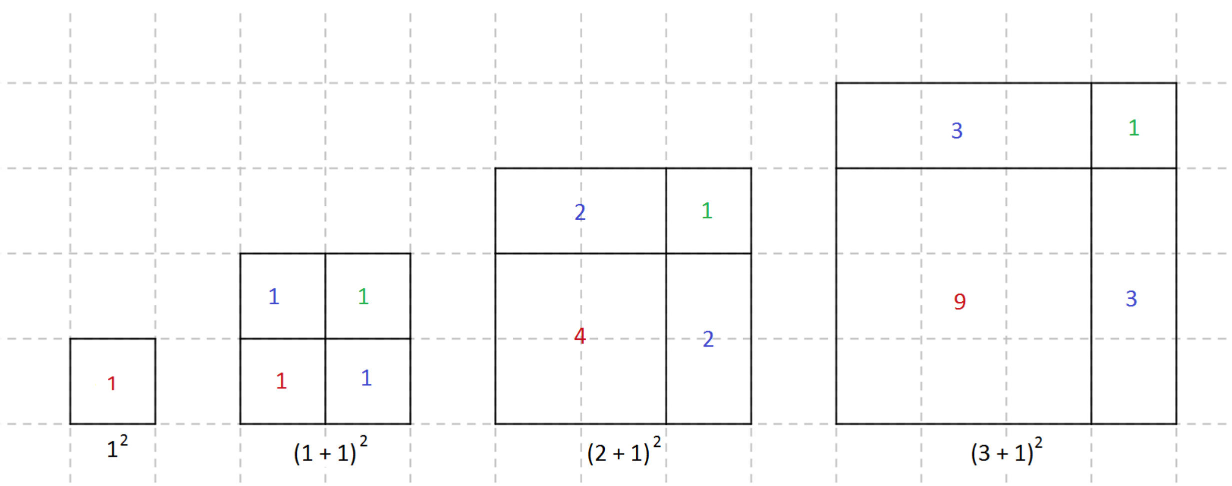
To manage their multiplication, the Babylonians turned to Geometry. We have already seen how an area can represent a multiplication:



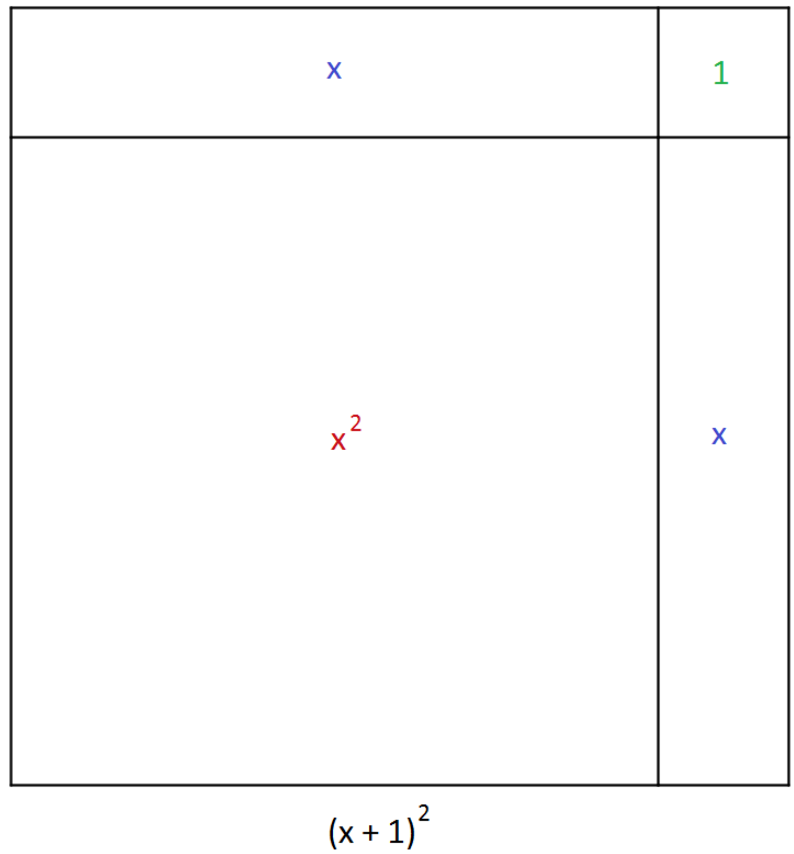
And we also saw how unknown quantities can be used in such a multiplication:



The Babylonians used square numbers to create a basic framework for their multiplications. They started at 1 and squared as many numbers as they could, creating a large table of squares. Here is how they would have started:



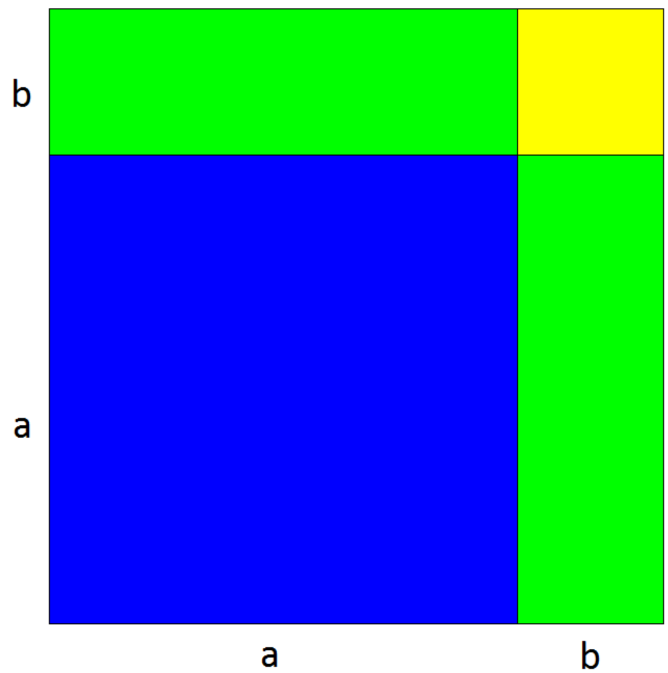
If you were making a table of the first 60 squares, you would soon notice a pattern:



This would tell you that $(x + 1)(x + 1) = x^2 + x + x + 1$, which is the same as $x^2 + 2x + 1$.

From here, it is not such a big leap to consider $(a + b)^2$, so let's recreate that discovery. Try to get some construction paper for this project, because it makes it much easier to see what is going on. We are going to actually square (a + b) by creating a square with sides (a + b). To be able to see things clearly, we'll make 'a' significantly larger than 'b'. Let's make a 6 inches long, and b 3 inches, which makes the best use of a sheet of construction paper, and allows us to match our construction with the one we'll do in the next chapter.

Starting at the bottom left corner of your paper, measure out length 'a' horizontally and vertically and make a small pencil mark at each location. Now you should have a square with sides a. Next add 'b' both horizontally and vertically to get a square with sides a + b. To complete this square you need to draw a straight line that is exactly perpendicular to the edge. You can use a protractor for this, or use your ruler to make a mark at the other edge of the paper. Then draw your line straight across. Cut off the strip of paper that you don't need, to end up with a 9" by 9" square. Once you have a square with sides a + b, divide it up as shown in the figure below. If you do the same with two additional sheets of paper you can have a nice colorful picture.

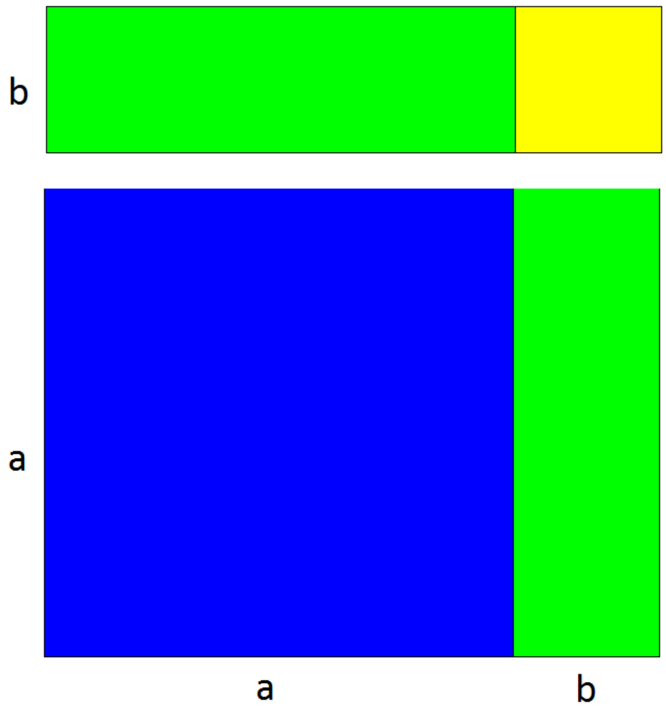


The blue square in the figure has sides a, so its area is a^2. The green rectangles have a longer side 'a' and a shorter side b. The area of each rectangle is ab. Since there are two of them, their total area is ab + ab, or 2ab. The little yellow square has sides b, so its area is b^2. From this, you can see that

$(a + b)^2 = a^2 + 2ab + b^2$

Save your square to use for the next chapter.

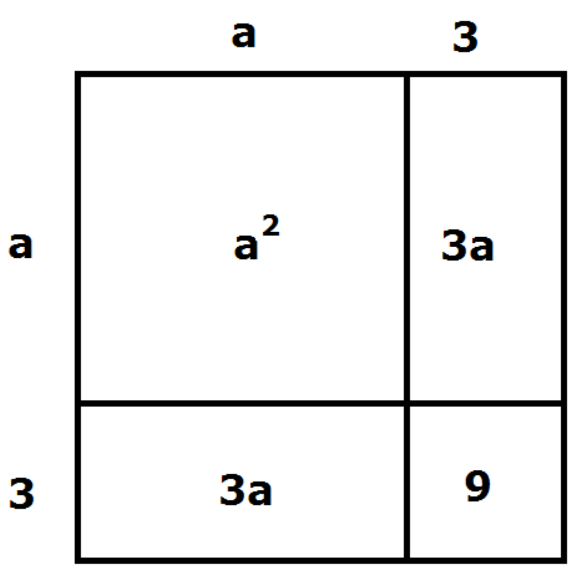
We don't always want to go to the trouble of constructing or drawing a square. There is also an easy way to square the sum of two numbers by using the distributive property. If you look at your square you can see that (a + b)(a + b) = a(a + b), which is the bottom part of the square, and b(a + b), which is the top part:



You can just multiply out $a(a + b)$ and $b(a + b)$ to get $a^2 + ab + ab + b^2$:

$(a + b)(a + b) = a(a + b) + b(a + b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$

So, if you need to square $(a + 3)$, you can draw a picture:



You can also multiply it out like this:

$(a + 3)^2 = (a + 3)(a + 3) = a(a + 3) + 3(a + 3)$

Use the distributive property to get $a^2 + 3a + 3a + 9$, which equals $a^2 + 6a + 9$.

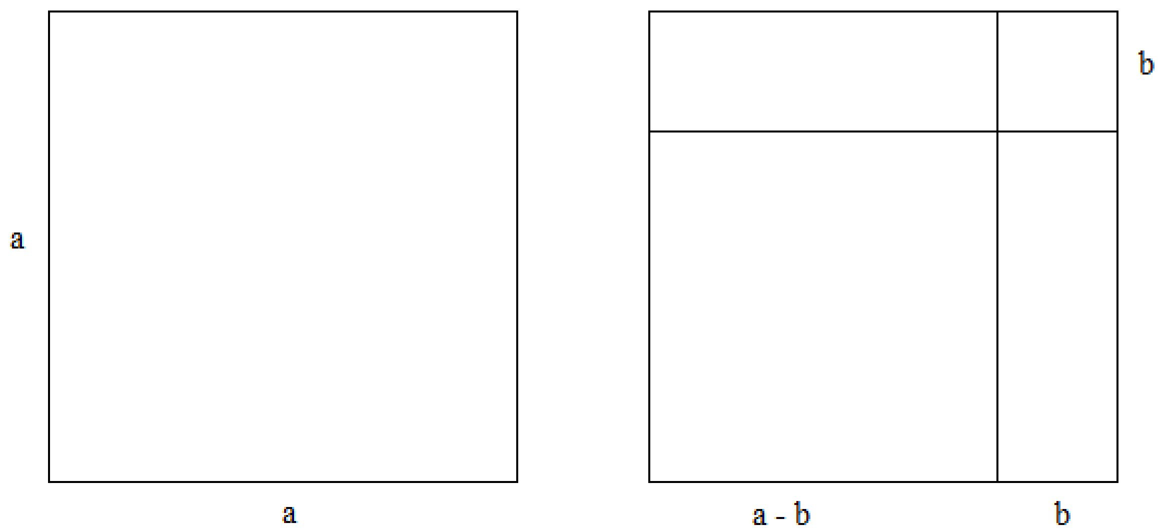
The Babylonians knew that the product ab of two numbers appears in the equation $(a + b)^2 = a^2 + 2ab + b^2$, so if you take $(a + b)^2$ and subtract a^2 and b^2 you are left with 2ab. If you want ab alone, you have to divide the whole thing by 2, like this: $ab = \frac{(a+b)^2-a^2-b^2}{2}$. To make it easier to do multiplications quickly, they used their tables of squares. Now all that people had to do to multiply two numbers a and b was to add them up and look up the values of $(a + b)^2$, a^2 and b^2 in a table. Then they used subtraction and divided the result by 2. To be fair, the average people on the street probably did not understand why this all worked, but hopefully they were grateful to their mathematicians for making their lives easier.

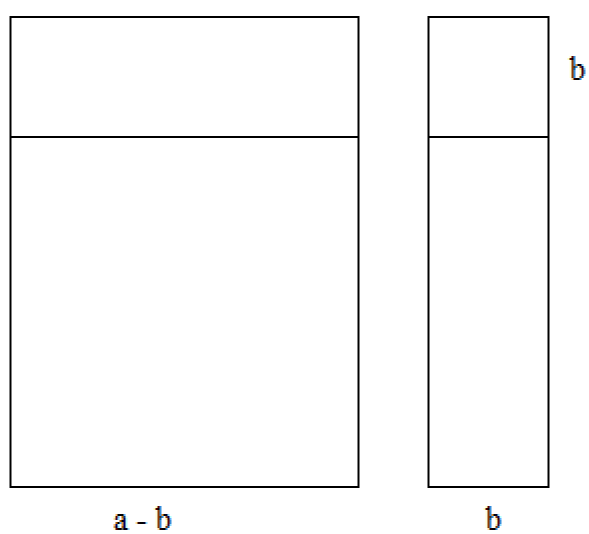
It is also possible to do $(a - b)^2$. Just follow along: $(a - b)(a - b) = a(a - b) - b(a - b)$. Now, you can look at this in two ways. We could do $a(a - b) - (b(a - b))$, which works but looks ugly because of all the parentheses. Usually people handle the second part by multiplying by $-b$. We take $-b(a - b)$ which is $-b \cdot a$ and $-b \cdot -b$. The first product turns out to be $-ab$, because a negative number times a positive is negative. The second product, $-b \cdot -b$ turns out positive, to give b^2. Let's do the whole thing all at once: $a(a - b) - b(a - b) = a^2 - ab - ab + b^2 = a^2 - 2ab + b^2$.

$(a - b)^2 = a^2 - 2ab + b^2$

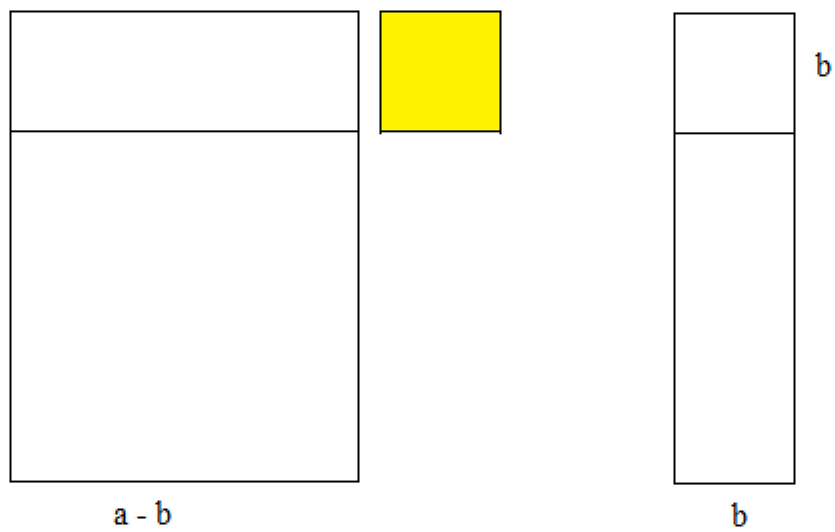
To illustrate this equation, we can draw a square with sides a, and inside of it a smaller square with sides a − b. It looks as though you can get the smaller square by taking the bigger one and cutting two strips away from it. Don't forget though that those two strips, each with length a and width b, overlap on the small square in the top corner, which is b^2. To be able to cut off two strips with area ab, you need to add an extra square with area b^2.

So $(a - b)^2 = a^2 - 2ab + b^2$. Try it out and make sure it works for you:





One strip with area ab has been cut off the original square (a^2), but now there is not enough paper left to remove another strip. The yellow square below represents an additional piece of paper with area b^2 that needs to be added in order for us to be able to take away a second strip with area ab:



Now you can cut away a second strip, so you have removed a total area of 2ab. Once you do that you will be left with a square that has an area of $(a - b)^2$.

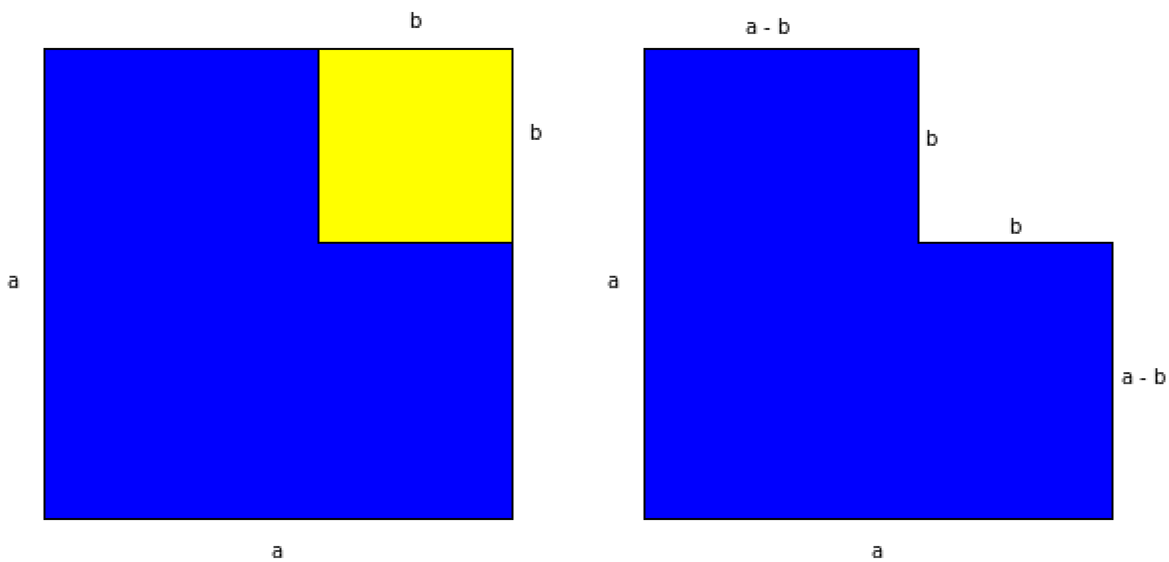
A more interesting product appears when we multiply $(a + b)(a - b)$.

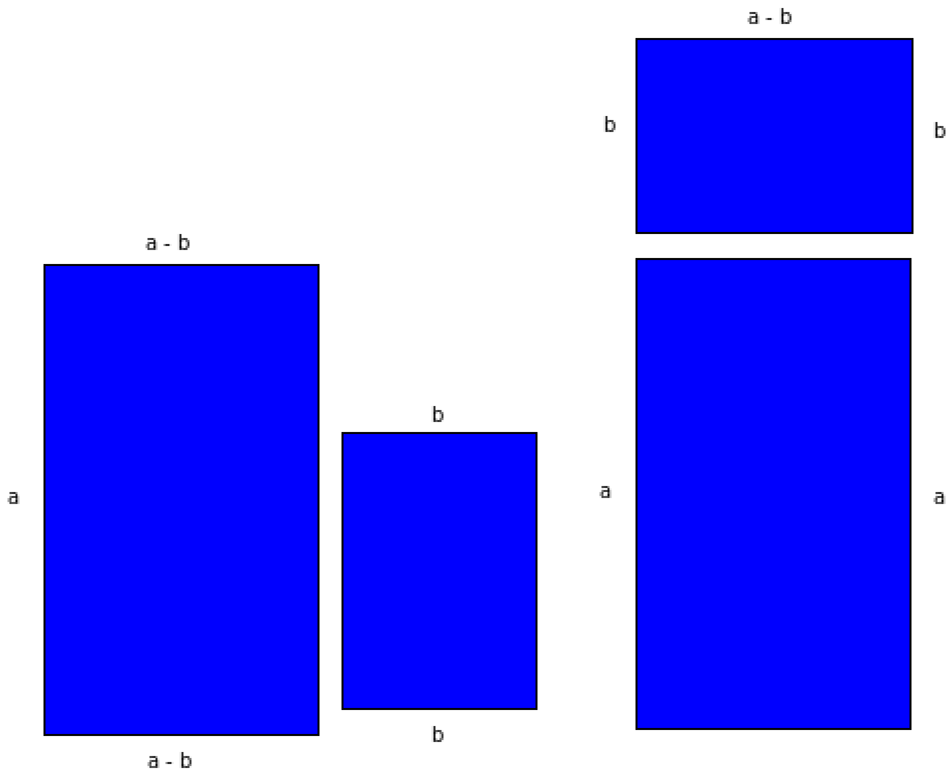
$(a + b)(a - b)$ can be written as $a(a - b) + b(a - b)$. Multiplying this out while carefully paying attention to the + and − signs, we get $a(a - b) + b(a - b)$, which is equal to $a^2 - ab + ab - b^2$. Notice that ab disappears altogether, and we are left with $a^2 - b^2$. This is called "the difference of two squares", and it is another favorite of people who construct algebra tests. You're pretty much guaranteed to see it on a college admission test. Usually you're expected to work this backwards and know that

$a^2 - b^2 = (a + b)(a - b)$

The actual problem is often cleverly disguised, like $x^2 - 16$, which is really $x^2 - 4^2$. That factors to $(x + 4)(x - 4)$.

Again, we can represent the difference of two squares with a paper square. Draw a square with sides a. On one corner of this square cut out a smaller square with sides b, which is the yellow area in the picture. Get rid of the small square. That is it already: $a^2 - b^2$. To see that this is the same as (a+b)(a–b), cut off one of the rectangular parts that is sticking out, and rotate it 1/4 turn. Now lay it against the remaining part to make a rectangle. The two sides of your rectangle are a + b and a – b, so its area is (a + b)(a – b).



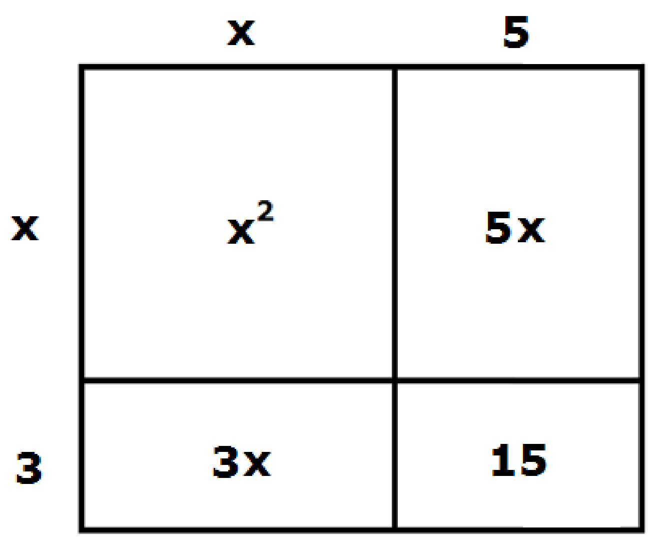


You can see that the area of the rightmost figure is (a + b)(a − b)

Note: At some point you may hear about something called FOIL. This stands for **F**irst, **O**uter, **I**nner, and **L**ast. It provides an arbitrary but consistent way to multiply the terms of polynomials. For a problem like (x + 5)(x + 3), you would multiply the first terms to get x^2, the outer terms to get 3x, the inner terms to get 5x, and the last terms to get 15. The acronym FOIL is only useful for polynomials that have two terms, and for people who speak English. It really does nothing to improve your understanding of multiplication, and the result is the same as you get by splitting up the first polynomial:

$(x + 5)(x + 3) = x(x + 3) + 5(x + 3) = x^2 + 3x + 5x + 15 = x^2 + 8x + 15$

Factoring (Chapter 16) will be easier to understand if you get used to writing out your multiplications instead of using FOIL. Some students prefer to use rectangles for their multiplications, like this:



Even if you don't need it now, you may want remember this method because it is actually quite useful for multiplications that have more terms. For example, $(x^2 + 6x - 1)(x^2 - 4x + 2)$ is easier to do with rectangles – try it out. Just put a minus sign in front of each negative number, and multiply carefully. You should end up with $x^4 + 2x^3 - 23x^2 + 16x - 2$.

SUMMARY

$(a + b)^2 = a^2 + 2ab + b^2$

$(a - b)^2 = a^2 - 2ab + b^2$

$(a + b)(a - b) = a^2 - b^2$

It is a good idea to memorize these 3 equations because they frequently appear on tests. Place them all in your permanent memory storage, right next to your name.

$(a + b)(c + d) = a(c + d) + b(c + d)$

Practice for More Multiplication

Here is a website that will help you practice your new skills:
http://www.mathguide.com/cgi-bin/quizmasters/PolynomialMult.cgi

Note that + signs are already filled in, so if you want a − sign you have to stick it in front of your number: x + 3 is the same as x + −3.

Babylonian Numbers

Read more about Babylonian math here:

http://www-history.mcs.st-and.ac.uk/history/HistTopics/Babylonian_mathematics.html

http://www-history.mcs.st-and.ac.uk/history/HistTopics/Babylonian_numerals.html

These pages are just for interest, and some of the information is rather difficult to understand without getting really involved in a somewhat outdated number system. Just read the parts you find interesting – no questions on the test.

Portfolio Chapter 15

Assignment 1

Now you know how to get $(a + b)^2$. But how can we do $(a + b)^3$? This expression means $(a + b)(a + b)(a + b)$, or $(a + b)(a^2 + 2ab + b^2)$. Using the same method we used before, it would seem reasonable to multiply like this:

$a(a^2 + 2ab + b^2) + b(a^2 + 2ab + b^2)$.

Hmm, that looks like a lot of work, so I'll leave it for you. Make sure to check your work using sample numbers for a and b.

Assignment 2

To find the product of two numbers a and b, the Babylonians also used the formula

$$ab = \frac{(a+b)^2-(a-b)^2}{4}$$

Show why this works by doing the multiplication and division to prove that it is true.

Next pick some real numbers for a and b and show how an average Babylonian would do a multiplication using this formula.

Chapter 16: Tricks with Triangles

"Geometry is knowledge of the eternally existent." – Pythagoras.

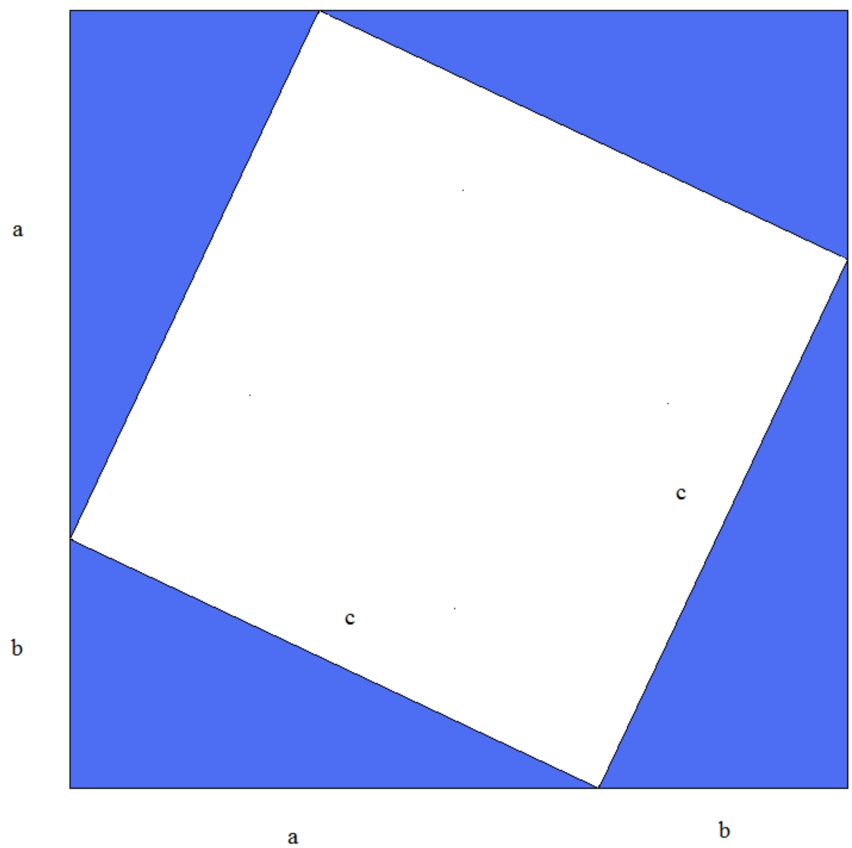
Mathematics > Patterns > Harmony > Perfection > Truth.

Pythagoras devoted most of his life to mathematics. He believed that reality is mathematical in nature, so if you want to know the truth about the universe you should study math. Many of his students lived with him in a community that we would probably consider to be a cult today. They followed strict rules of conduct, and kept mathematical discoveries secret.

Let's look at one of these discoveries, which fortunately is no longer a secret. We do not know if this particular discovery was made by Pythagoras himself, or by one of his students, since it was common in those days to attribute all big accomplishments to the master rather than the student.

Use construction paper for this project if it is available. We will also need the large (a + b) square that you created earlier. Start by drawing a triangle that has two **perpendicular** sides, a and b. Use your protractor to get the two sides at exactly a 90 degree angle. [If you are not sure about degrees, check out this website: http://www.mathsisfun.com/geometry/degrees.html.] Make side a 6" long, and side b 3" long. Now when you go to complete the triangle by drawing the last side, c, you'll see that you do not get a choice for the length of c. Pythagoras apparently drew quite a few of these triangles that contain one 90 degree [or "right"] angle, and he realized that once you picked the lengths for two of the sides, the length of the third side is already determined. There had to be some formula for the relationship between the lengths of these three sides, but what was it?

Because the Pythagoreans were so secretive, we cannot be sure how they eventually discovered the formula. The best we can do is try to re-create what they might have done. They were interested in harmony and perfection, so if they were studying something with a 90 degree angle they might have gathered four of these things to have a perfect 360 degrees. Create 3 more triangles that are identical to the first one you made. Arrange them as shown in the picture to create a square with sides (a + b).

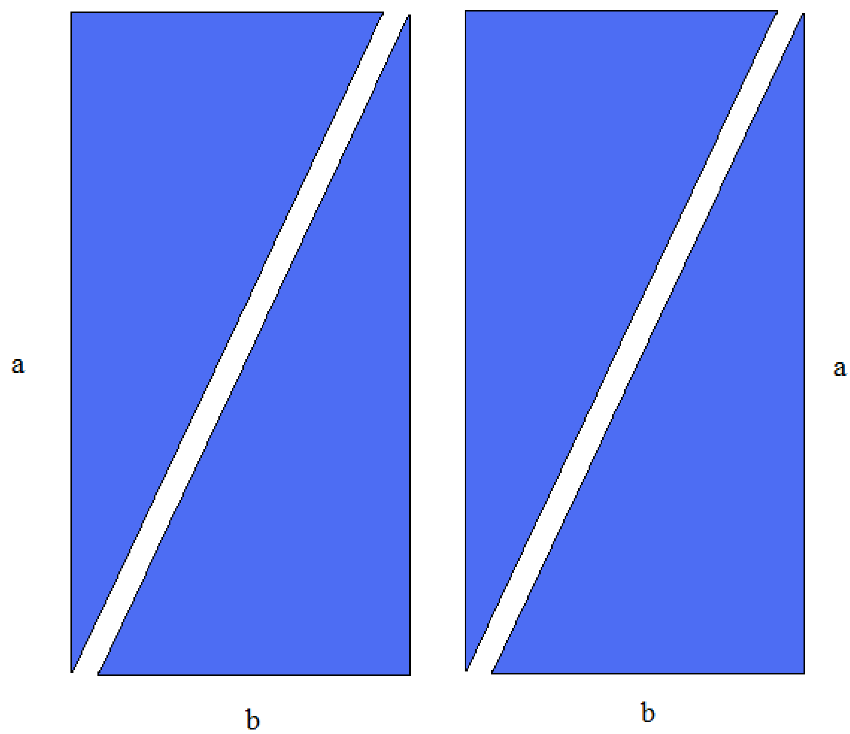


Now consider the following questions:

1. What is the area of this figure? Could it possibly have the same area as the large square you made in the previous chapter? Put the two shapes next to each other to compare them.

2. Consider the empty space inside the shape you made with the triangles. What is its area? Use white construction paper to make a square that fits in the empty space.

3. Take apart the shape you made with the triangles. Put the triangles together in pairs to make two rectangles. What is the total area of these two rectangles?



4. Put the triangle shape back together, with the white square in the empty space. Now you have a big square. What is left if you remove a total area of 2ab from this square?

5. Look at the other square that is made out of squares and rectangles. Remove a total area of 2ab from this square also. What is left?

6. Both squares had the same area to start with. You removed 2ab from the first square, and 2ab from the second square. Can you create an equation with the parts that are left? How do you know these parts are equal?

7. Look at the equation you have discovered. Is it true for any right triangle with sides a, b and c where c represents the **hypotenuse** [the side opposite the right angle]? To test this out, draw some random right–angled triangles. Particularly convenient values to use for a and b are 3 and 4.

The formula you just discovered is the **Pythagorean Theorem**. It gives the relationship between the three sides of a right triangle, and it is very important. Memorize it, because questions about it are sure to be found on college admission tests.

There are many more interesting things to learn about triangles, but those belong in geometry and trigonometry so you'll see them later.

Pythagoras and his Followers

Read the Wikipedia article about Pythagoras here: http://en.wikipedia.org/wiki/Pythagoras

Practice for the Pythagorean Theorem

Here are some practice problems: http://www.regentsprep.org/Regents/math/geometry/GP13/PracPyth.htm

Portfolio Assignment

Find the shortest distance between the point (15,7) and the line $y = -\frac{3}{4}x + 12$.

This is a challenging problem. You now have the knowledge required to solve it, but you may have to do some experimenting. What exactly is the shortest distance between a line and a point? Take out a ruler and do some measuring. Draw a line segment (a part of an infinitely long line) where you find the shortest distance. How will you find the intersect point, which is the point on the line that is closest to (15,7)? Once you have two points, how can you find the distance between them?

Chapter 16 Quiz

Question 1

The lengths of two of the sides of a triangle are 3 cm and 4 cm. The angle between these two sides is 90 degrees. This is a right triangle. What is the length of the hypotenuse (the longest side)? _______ cm

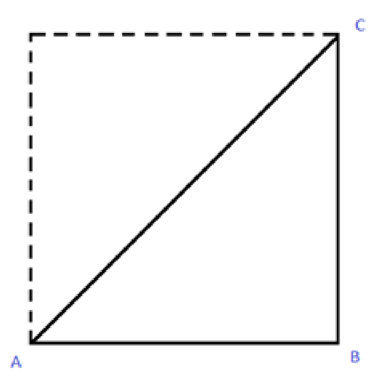
Question 2

The Pythagorean theorem says that if a triangle is a right triangle, $a^2 + b^2 = c^2$, where c is the hypotenuse. The converse of this statement would be that if $a^2 + b^2 = c^2$ then the triangle is a right triangle. The converse of a statement is not necessarily true. For example, if I want to check my e–mail, I go to my computer. Converse: If I go to my computer, I want to check my e–mail – not necessarily true because I may want to play a game or read about math online.

So is the converse of the Pythagorean Theorem true? Grab a ruler and a protractor and check it out for yourself:

A triangle has sides of length 6 inches, 8 inches, and 10 inches. Is it a right triangle? _________

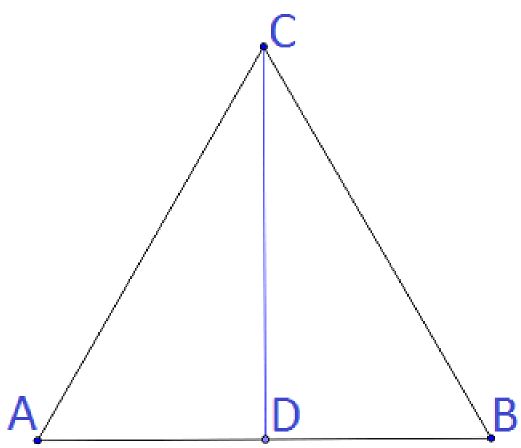
Question 3



Triangle ABC shown above is a special kind of right angled triangle. Side AB is equal in length to side BC. This is the kind of triangle you get when you cut a square diagonally. The angles are 45°, 90°, 45°.

If the length of side AB is 1 inch, what is the length of the hypotenuse, AC? _______ inches

Question 4



The triangle ABC is an equilateral triangle. All its sides are the same length. The blue line from point C to point D divides this equilateral triangle into two triangles that are exact mirror images of each other. Because angle ADC and angle BDC are equal and add up to 180 degrees, they are each 90 degrees.

If the length of side AC is 2 inches, find the length of the blue line CD. _______ inches

Chapter 17: Factoring

By now, you are good at solving equations with one unknown, like $3x - 5 = 10$. However, when that unknown has an exponent associated with it things can get a bit more complicated. Take, for example, $x^2 + 6x + 8 = 0$. Now it is not so easy to figure out the value of x, even though it is the only unknown in the equation. An equation that has x^2 [x squared], as the term with the highest power is a quadratic equation. The ancient Babylonians already ran into such equations because they were working with squares. The root "quadr" means 4 and "quadrus" means square, since a square has four sides. Quadratic equations are important in mathematics, and also in science. For example, the position of a falling object at time t is given by the equation $P = -4.8t^2 + v_it + P_i$, where v_i is the initial speed and P_i is the initial position. The general form of a quadratic equation is:

$ax^2 + bx + c = 0$

Here a, b and c are actually numbers that depend on the particular equation you are trying to solve. $x^2 + 6x + 8 = 0$ is a quadratic equation where a is just 1, so it doesn't show, b is 6, and c is 8. The numbers b and c can be zero. $4x^2 - 36 = 0$ is also a quadratic equation, but because b is zero the middle term is missing. When c is zero you see only the first two terms, as in $3x^2 + 5x = 0$.

Quadratic equations usually have two answers for the value of x, which makes guessing at the answer even more useless. Some quadratic equations can be solved by taking the sum $ax^2 + bx + c$ and re-writing it as a product. This helps a lot, because if the product of two numbers is 0, that means that one of those numbers, or both, have to be zero. Just think about it. If $n \cdot m = 0$, the only way that can be true is if either n or m, or both, are zero. The process of rewriting the sum as a product is called **factoring**.

Taking out the Greatest Common Factor

The first thing you should do if you are asked to factor something is to see if all of

the terms have some factor in common. Taking out a common factor is like using the distributive property backwards. If 3(x + 4) = 3x + 12, then we can take 3x + 12 and factor it to get 3(x + 4).

As an example, we will try to solve $10x^2 + 5x = 0$ by factoring. The largest factor that $10x^2$ and 5x have in common is 5x, so you can rewrite the equation:

$10x^2 + 5x = 0$

$5x(2x + 1) = 0$

There, we have changed $10x^2 + 5x = 0$ into $5x(2x + 1) = 0$. In this case you are multiplying 5x and 2x + 1 to get zero, so either 5x = 0 or 2x + 1 = 0. This gives us two answers for the value of x. If 5x = 0 then x must be 0. If 2x + 1 = 0 then 2x = -1 and $x = -\frac{1}{2}$. Plug those answers into the equation to see that they are both correct:

If x = 0 then $10(0)^2 + 5(0) = 0$.

If $x = -\frac{1}{2}$ then you get $10\left(-\frac{1}{2}\right)^2 + 5\left(-\frac{1}{2}\right)$. Since $\left(-\frac{1}{2}\right)^2$ is $-\frac{1}{2} \cdot -\frac{1}{2} = \frac{1}{4}$, the equation now says that $\frac{10}{4} + -\frac{5}{2} = 0$, which is true.

Remember, no matter which factoring method you use, always see if you can take out a common factor first! This is particularly important when there is an x^3 in your problem, because all of the other methods are for quadratics. If you see something like $3x^3 + 6x^2 + 12x$, you should notice that all of the numbers can be divided by 3, and all of the terms have an x in them:

$3x^3 + 6x^2 + 12x$

$3x(x^2 + 2x + 4)$

Simple Factoring

For an equation like $x^2 + 6x + 8 = 0$, there is no common factor that we can take out. Now how can we solve this?? Well, many quadratic equations are the product of two simple factors, like $(x + p)(x + q)$, where p and q can be either positive or negative numbers. To see how this works, do the following multiplications:

$(x + 3)(x + 1)$

$(x + 2)(x + 4)$

$(x - 5)(x - 6)$

$(x + 3)(x - 2)$

In each case, the result is a quadratic equation of the form $x^2 + ...x + ...$ Usually mathematicians write these general equations as $x^2 + bx + c$. The constant a is the number 1 for these simpler quadratic expressions. If you look closely at your answers, you'll see that b, the number in front of the middle term, is always the sum of whatever numbers we picked for p and q, while c is always the product of these 2 numbers. For example, $(x + 3)(x - 2) = x(x - 2) + 3(x - 2)$, which works out to $x^2 + 3x - 2x - 6$, and simplifies to $x^2 + x - 6$. Here b, the middle number in front of the x, is 1, and c is -6. The value of b is determined by adding 3 and -2, and the -6 at the end is caused by multiplying 3 and -2.

Watch what happens when we multiply just using p and q:
$(x + p)(x + q) = x(x + q) + p(x +q) = x^2 + qx + px + pq$

This can be written as $x^2 + (p + q)x + pq$. That's always how things work out. The middle term ends up as the sum of p and q [times x], and the last term is the product pq.

For the equation $x^2 + 6x + 8 = 0$ we need to look for p and q such that p + q is 6, and pq is 8. The only two numbers that fit this description are 2 and 4. It doesn't matter which one we call p and which one we call q, because $(x + 2)(x + 4)$ is the same as $(x + 4)(x + 2)$.

If $(x + 2)(x + 4) = 0$, then either $(x + 2) = 0$, or $(x + 4) = 0$. This gives us two answers for the value of x:

$x + 2 = 0$, so $x = -2$, or

$x + 4 = 0$ so $x = -4$.

Plug those answers into the equation to see for yourself that they are both correct.

There is an interesting way to actually see these answers. We can create the function $y = x^2 + 6x + 8$. This gives many points (x,y) for which the equation is true. At the point where $y = 0$ we should find our answers $x = -2$ and $x = -4$. Check this out by drawing the graph. The easiest points to plot are for $x = 1, 0, -1, -2, -3$ and -4. When x is 2, y will be $1^2 + 6 \cdot 1 + 8$, which is 15. When $x = 0$, $y = 8$, and so on. Connect your calculated points with a smooth curved line. At the points where $y = 0$ the graph crosses the x–axis, and you should be able to see your answers here. You can label these particular points. Underneath the graph you can write the equation and show how you factor it to obtain the answers. This would be a good page for your portfolio.

To check your work, use MathGV, Desmos online graphs, or some other graphing program. Enter y = x^2 + 6x + 8.

At this point you should stop to realize that while it is very helpful to rewrite $x^2 + 6x + 8 = 0$ as $(x + 2)(x + 4) = 0$, it would not at all be useful to rearrange an equation like $x^2 + 6x + 8 = 3$ into $(x + 2)(x + 4) = 3$. Here you cannot say that either $x + 2$ must be 3, or $x + 4 = 3$. That trick only works with zero! To solve $x^2 + 6x + 8 = 3$, first rewrite it as $x^2 + 6x + 5 = 0$, and then factor it.

Finding The Two Numbers

Many quadratic equations have negative terms. When you look for your two numbers that add to the middle number and multiply to the last number, there are a lot of possibilities! Although it makes sense to try a guess first, if you can't see the solution quickly you need a system that will work reliably.

1. Decide if you are looking for two positive numbers, two negative numbers, or one positive and one negative number.

For example, if two numbers multiply to 15 and add to –8, they would have to both be negative to make that work. If the numbers multiply to –60, there has to be one positive and one negative. If they multiply to 100 and add to 25, they must both be positive.

2. If one number is positive and the other is negative, decide which one should be "larger".

If you want a positive sum, the positive number must be larger than the negative one. If the sum should be negative, pick the larger number and give it a minus sign.

3. Work systematically.

Divide the product by all possible factors in turn. Always start with 1, so you don't overlook this somewhat obvious possibility! Next, remember that a number is only divisible by 3 if the sum of its digits is divisible by 3. Also, if you divide it by 2 and the result is even, you can divide it by 4. If you divide it by 3 and the result is even, it is also divisible by 6, and so on. Make sure to also test prime numbers like 7, 11, 13, 17, 19 etc, using a calculator if necessary. Eventually you will reach a halfway point, after which the numbers repeat in reverse order, so you can stop. If you don't have the solution by then, you have either overlooked it, or it doesn't exist.

Example

Solve for x: $x^2 - 2x - 120 = 0$

Because the last number is negative, I know to look for a positive and a negative number. Those numbers have to add to −2, the middle number, so I want to take the larger of those two numbers and give it a minus sign. I can also see that the two numbers must be close in value because they add to such a small number even though they have quite a large product. I will look for them systematically by considering which two numbers can multiply to 120. To accomplish that I always start with 1, and then follow the divisibility rules. 120 is even, so it can be divided by 2. To see if a number is divisible by 3, add up its digits and check if the sum is divisible by 3. For 120 that is 1 + 2 + 0 = 3, which is divisible by 3. Next, I keep watching for even numbers. When I divided 120 by 2 the result was still even, so it must be also divisible by 4. If a number ends in 5 or 0 it must be divisible by 5, and 120 is 5 times 24. I divided by 3 and got an even number, so I can divide by 6. Then I tried 7 on my calculator, but it didn't work. Division by 8 works because division by 4 gives an even number. Nine will not work because if you divide by 3 the result, 40, is not divisible by 3. Ten works because 120 ends in 0.

$$\begin{aligned} 120 = \ & 1 \cdot 120 \\ & 2 \cdot 60 \\ & 3 \cdot 40 \\ & 4 \cdot 30 \end{aligned}$$

$5 \cdot 24$

$6 \cdot 20$

$8 \cdot 15$

$10 \cdot 12$

Those last two numbers are close enough together to have a chance of adding up to -2, and they do if I put the minus sign on the larger number, 12. The two numbers I want are 10 and -12:

$(x + 10)(x - 12) = 0$

$(x + 10) = 0$	or	$(x - 12) = 0$
$x = -10$	or	$x = 12$

Factoring Practice

Go back to http://www.shodor.org/interactivate/activities/AlgebraQuiz/, and this time check the box that says quadratic. Make sure it is set to "level 1" with no time limit. We'll do the hard ones later.

Here is somewhere you can go to just factor, without worrying about producing any actual solutions. Make sure to select "Monic Polynomials" and "Degree 2".
http://www.saab.org/mathdrills/factor.cgi

The Difference of Two Squares

Sometimes your problem doesn't look like the standard quadratic equation, such as $x^2 - 9 = 0$. This equation can be solved by writing it as $x^2 = 9$, which means that x is 3 or $x = -3$. It can also be solved by factoring. We do this by using the difference of two squares. You may want to review the chapter “More Multiplication”, where we saw that $(a + b)(a - b) = a^2 - b^2$. Anything that looks like the difference of two squares can be factored into $(a + b)(a - b)$. Don’t get confused; a and b here do not have anything to do with the a and b in $ax^2 + bx + c$. People use a and b frequently because they just happen to sit conveniently at the beginning of the alphabet. $x^2 - 9$ is really $x^2 - 3^2$, and you can factor it to solve a quadratic equation. Simply replace a with x, and b with 3:

$a^2 - b^2 = 0$

$(a + b)(a - b) = 0$

$x^2 - 9 = 0$

$(x + 3)(x - 3) = 0$

Multiply $(x + 3)(x - 3)$ back out to see that it really is $x^2 - 9$

If $(x + 3)(x - 3) = 0$, your answers are $x = -3$ and $x = 3$. It is a good idea to create another graph, $y = x^2 - 9$, to check these answers.

Another way to look at $x^2 - 9$ is to write it as $x^2 - 0x - 9$. Now you could factor it by looking for two numbers that multiply to -9 and add to 0. Those numbers are 3 and -3, so the answer would still be $(x + 3)(x - 3)$. Even if you find this approach easier, you should still know the difference of two squares for more complex situations.

The difference of squares may be cleverly disguised by multiplying everything by a random number. Just take something like $x^2 - 25 = 0$ and multiply both sides by 3 to get $3x^2 - 75 = 0$. Now it is harder to see that you can factor it as the difference of two squares, but if you know to watch for this trick it isn't so bad. Just factor out the 3:

$3x^2 - 75 = 0$

$3(x^2 - 25) = 0$

$3(x + 5)(x - 5) = 0$

Divide both sides by 3:

$(x + 5)(x - 5) = 0$

$x = 5$ or $x = -5$.

Factoring by Grouping

Sometimes a quadratic equation will be presented to you with the middle term split up, like this: $x^2 + 4x + 5x + 20 = 0$. This is a subtle hint that you're expected to factor the equation by grouping. We can do that by taking the four terms on the left side of the equation and putting them in two groups:

$$x^2 + 4x + 5x + 20 = (x^2 + 4x) + (5x + 20)$$

Now each group of terms has a common factor. Always make sure you take out the largest possible thing that the terms have in common:

$$x(x + 4) + 5(x + 4)$$

Because you have x times (x + 4) *and* 5 times (x + 4), you can change that to

$$(x + 5)(x + 4)$$

Check your work by multiplying the factors back out:

$$(x + 5)(x + 4) = x(x + 4) +5(x + 4) = x^2 + 4x + 5x + 20 = x^2 + 9x + 20$$

So, the equation $x^2 + 4x + 5x + 20 = 0$ factors to $(x + 5)(x + 4) = 0$, and x has to be −5 or −4. You'd get these same answers if you had added the middle terms up first, since $x^2 + 4x + 5x + 20 = 0$ is the same as $x^2 + 9x + 20 = 0$ Using the standard method you'd look for two numbers that add up to 9 and multiply to 20, and again you get $(x + 5)(x + 4) = 0$

Some people prefer a more visual method to handle grouping. Starting with $x^2 + 4x - 5x - 20$, draw a square and place the four terms inside, making sure that the last term is diagonally opposite the first:

x^2	$-5x$
$4x$	-20

Next, take out the greatest common factors both vertically and horizontally, in the same way that we did before. Place the factors above and beside the square like this:

	x	-5
x	x^2	$-5x$
4	$4x$	-20

Notice that each little square is now the product of the terms on top and to the right: x^2 is the product of x and x, -5x is the product of x and -5, 4x is the product of 4 and x, and -20 is the product of -5 and 4. Add up the parts outside the square to get (x + 4)(x – 5).

Try this method with the other problems shown in this section, so you can see if you like it better.

Here is another example:

$(6x^2 + 2x) + (15x + 5) = 2x(\mathbf{3x + 1}) + 5(\mathbf{3x + 1}) = (2x + 5)(\mathbf{3x + 1})$

The order of the terms doesn't make a difference in how things turn out:

$(6x^2 + 15x) + (2x + 5) = 3x(\mathbf{2x + 5}) + 1(\mathbf{2x + 5}) = (3x + 1)(\mathbf{2x + 5})$

Notice that the things marked in green must be the same in both sets of parentheses. If they don't match, you have likely done something wrong.

Factoring by grouping is a little trickier if placing the parentheses could potentially mess up the order of operations:

$12x^2 + 10x - 18x - 15 = 0$

When you place the second set of parentheses, you have to account for the fact that both 18x and 15 are being subtracted. Change the sign as shown in green:

$(12x^2 + 10x) - (18x \mathbf{+} 15) = 0$

$2x(6x + 5) - 3(6x + 5) = 0$

$(2x - 3)(6x + 5) = 0$

$2x - 3 = 0$ or $6x - 5 = 0$

$2x = -3$ or $6x = 5$

$x = -\frac{3}{5}$ or $x = \frac{5}{6}$

If those little minus signs keep tripping you up, you can sometimes avoid them by switching the terms around:

$12x^2 + 10x - 18x - 15 = 0$

$12x^2 - 18x + 10x - 15 = 0$

$(12x^2 - 18x) + (10x - 15) = 0$

$6x(2x - 3) + 5(2x - 3) = 0$

$(6x + 5)(2x - 3) = 0$

$x = -\frac{3}{5}$ or $x = \frac{5}{6}$

The ac Method

There may be a number in front of x^2, as in $2x^2 + 10x + 8 = 0$. The first thing to do is to see if you can get rid of it. In this particular case you can divide the entire equation by 2 to get $x^2 + 5x + 4 = 0$. This factors as $(x + 4)(x + 1) = 0$, and the answers are −4 and −1. If there is no equals sign, and you are just asked to factor $2x^2 + 10x + 4$, put the 2 in front: $2x^2 + 10x + 4 = 2(x + 4)(x + 1)$.

Potentially confusing is the presence of a minus sign in front of the x, as in $-x^2 + 6x - 9$. Fortunately the minus sign is easy to remove; just factor out −1 to turn that into $-1(x^2 - 6x + 9)$. Then factor the part in the parentheses to get $-1(x - 3)(x - 3) = -(x - 3)^2$.

$-x^2 + 6x - 9 = -(x^2 - 6x + 9)$

If you find a number in front of x^2, you usually can't remove it. When I was young we had to factor such equations by guessing, which took a long time even if the numbers were small. That was a real pain, and there is actually a better way.

When there is a number in front of x^2, how did it get there? If you just multiply something like $(x + 3)(x + 4)$, you end up with $x^2 + 7x + 12$. To create some number other than 1 in front of x^2, at least one of the factors must have a number in front of its x. Let's try $(2x + 3)(x + 4)$. [Even though 2 doesn't make a good sample number when you are checking answers, I often use it in experimental math to keep things simple. There shouldn't be a problem so long as there is only one 2 present.] Anyway, $(2x + 3)(x + 4) = 2x(x + 4) + 3(x + 4) = 2x^2 + 8x + 3x + 12$. If you leave it like that it is easy to factor, because you can just work backwards:

$2x^2 + 8x + 3x + 12$

$(2x^2 + 8x) + (3x + 12)$

$2x(x + 4) + 3(x + 4)$

$(2x + 3)(x + 4)$

Unfortunately, once you simplify $2x^2 + 8x + 3x + 12$ to $2x^2 + 11x + 12$, it becomes very difficult to factor. We don't know what caused the middle term to be 11x. It could have been 8x + 3x, or x + 10x, or 5x + 6x. You can try factoring $2x^2 + x + 10x + 12$ or $2x^2 + 5x + 6x + 12$ by grouping, but you'll see that it doesn't work. And if the middle number was larger, there would be even more incorrect ways to split it up. If only we could somehow know that the middle number 11 was caused by adding 8 and 3. Well, maybe we can.

I do not know who discovered the ac method, but I am very grateful to this person for naming it in a way that tells you how to use it. ac means a times c, and what we need to do for this to work is to multiply a and c for any quadratic that looks like $ax^2 + bx + c$. For $2x^2 + 11x + 12$, a is 2 and c is 12. So, ac is 24. Now all you have to do is find two numbers that multiply to 24, and add to 11, the middle term. Those numbers are 8 and 3. Unlike with simple factoring, those numbers don't immediately give us the answer, but once we have them we can use them to **split the middle term**, which stretches the quadratic expression out into four terms. Just rewrite it as $2x^2 + 8x + 3x + 12$. Now that there are four terms you can factor by grouping:

$2x(x+4) + 3(x +4)$

$(2x + 3)(x + 4)$

The ac method also works for quadratic expressions where a is 1, such as $x^2 - 7x + 12$. You can look for two numbers that multiply to 12 and add to −7, and use those numbers to split the middle term like this: $x^2 - 3x - 4x + 12$. Group the terms, paying close attention to the signs: $(x^2 - 3x) - (4x - 12)$, and then factor: $x(x - 3) - 4(x - 3) = (x - 4)(x - 3)$. It works, but it's a bit like calling in the wrestling team to help you move your desk. You can get the job done faster, and without people making fun of you, by just using simple factoring: $x^2 - 7x + 12 = (x - 3)(x - 4)$.

Although most students just worry about how to use the ac method, you should really think about why it works, and how it was discovered.

Let's look a little more closely at the multiplication $(2x + 3)(x + 4)$:

$(2x + 3)(x + 4)$

$2x(x + 4) + 3(x + 4)$

$2 \cdot x^2 + 2 \cdot 4 \cdot x + 3 \cdot x + 3 \cdot 4$

$2x^2 + 8x + 3x + 12$

So where do those two middle numbers come from? The 8 came from $2 \cdot 4$, and then we have a 3. Notice that you can see those same numbers, 2, 3 and 4 when you look at the front and the back of the quadratic expression:

$(2x + 3)(x + 4)$

$2x^2 + 2 \cdot 4\, x + 3x + 3 \cdot 4$

If I take the 2 in front of the x^2, and multiply it by the numbers at the end, I get $2 \cdot 12$, or $2 \cdot 3 \cdot 4$. There is an 8 in there, which is $2 \cdot 4$, and a 3 also. The middle number, 11, is $2 \cdot 4 + 3$. So, if I search for two numbers that multiply to 24 [a times c] and add to 11, I should end up with the correct two numbers to split the middle term.

Try it out on the original expression:

$2x^2 + 11x + 12$

Multiply $2 \cdot 12$, which is 24. The number 24 "contains" the numbers that add to 11. Only 8 and 3 multiply to 24 and add to 11. Now we can split the middle term, and then factor by grouping.

Once you see how something works, you can try it out on slightly harder problems, like maybe $(3x + 7)(5x + 6)$. That multiplies to $3x(5x + 6) + 7(5x + 6) = 15x^2 + 18x + 35x + 42 = 15x^2 + 53x + 42$. Again, look at it more closely:

$(3x + 7)(5x + 6)$

$3x(5x + 6) + 7(5x + 6)$

$3 \cdot 5 \cdot x^2 + 3 \cdot 6 \cdot x + 7 \cdot 5 \cdot x + 7 \cdot 6$

$15x^2 + 18x + 35x + 42$

$15x^2 + 53x + 42$

The numbers that we need are 18, which came from $3 \cdot 6$, and 35, from $7 \cdot 5$. If I take the number in front of the x^2, which is $3 \cdot 5$, and the last number, $7 \cdot 6$, and multiply them, I get $3 \cdot 5 \cdot 7 \cdot 6$. This number "contains" both 18 and 35. To factor the expression $15x^2 + 53x + 42$, multiply 15 and 42 to get 630. Now all you need to do is find two numbers that add to 53 and multiply to 630. That's a bit time consuming, but not too bad if you have a calculator. You can divide 630 by 1, 2, 3, 5, 6, 7, 9, 10, 14, 15, 18, …, and by the time you get to 18 you have the numbers you are looking for.

Perfect Squares

Occasionally you can recognize an expression or equation as being a perfect square, which can save you some time. Recall from the chapter "More Multiplication" that $(a + b)^2 = a^2 + 2ab + b^2$. Again, these a's and b's are completely separate from the constants in the general quadratic equation. For example, $(x + 3)^2 = (x + 3)(x + 3)$, which multiplies out to $x^2 + 6x + 9$. If you see this expression you may recognize it as a perfect square. If not, you can still just factor it by simple factoring.

In general, perfect squares usually show up as a specific set of problems in a textbook. Just remember that anything that looks like $a^2 + 2ab + b^2$ can be changed to $(a + b)(a + b)$. Let's look at an example: $25x^2 + 20x + 4 = 0$.

$25x^2$ is a square because it is $(5x)^2$. 4 is also a square, so this is potentially a perfect square. If a is 5x, and b is 2, then the middle term, 2ab, should be $2(5x)(2) = 20x$. Because the middle part checks out, this is in fact a perfect square:

$25x^2 + 20x + 4 = 0$

$(5x + 2)(5x + 2) = 0$

$(5x + 2)^2 = 0$

Factoring a quadratic equation with large numbers in it can be quite a bit of trouble. Also, if you try making up some random quadratic equations yourself you'll quickly discover the real truth: Most quadratic equations cannot be solved by factoring! Fortunately there is another way, and we'll cover that in the next chapter. Alas, it is not simpler.

As we have seen an obvious purpose of factoring is to solve quadratic equations. We either want to find a value for x that would make the equation true, or we have a graph of $y = ax^2 + bx + c$ and we want to know where this graph touches the x axis [which would happen when $y = 0$]. Also, carefully designed problems that you will encounter later on will have you use factoring to simplify expressions like $\frac{x^2-4}{x-2}$. You will factor the top into $(x + 2)(x - 2)$, and then divide the top and bottom by $x - 2$, provided that x is not 2 so that you won't divide by zero.

Of course, that only works for special quadratics that can be factored. There seems to be something more going on here. Pay careful attention when you take exams outside this course, and you will see that you are usually asked to factor expressions rather than equations. This of course does not produce any sort of answers. In fact, all it produces is factors. Students in the United States spend a huge amount of time factoring, and they consistently outperform European students on tests requiring factoring. All I found on an internet search about the purpose of factoring was a math teacher complaining that his students had to spend four months doing nothing but factoring. What is the true reason for this? There can really be only one answer: It is all part of a secret CIA project. In the unlikely event of a foreign invasion of America, you will receive specially encoded alpha–numeric messages from your government. By factoring these messages and changing the numbers into letters, you will know exactly what is going on, while the poorly educated enemy will be completely clueless. Therefore, in the interest of national security, you need to practice factoring until you know it really well...

By the way, the more factors you can find the better. The question will often state: "Factor completely." If you see that the whole expression can be divided by a single number, like $12x^2 + 34x + 10$, do that first. $12x^2 + 34x + 10 = 2(6x^2 + 17x + 5) = 2(3x + 1)(2x + 5)$.

Factoring Summary

First: Take out a common factor

Always check first to see if you can remove a common factor:

$25x^2 + 10x = 5x(5x + 2)$.

$x^3 + 8x^2 + 16x = x(x^2 + 8x + 16)$

$-x^2 + 4x - 4 = -1(x^2 - 4x + 4)$

Factor by Grouping

Specially designed polynomials can be factored by grouping. There are usually 4 terms:

$3x^3 - 12x^2 - 4x + 16 = (3x^3 - 12x^2) - (4x - 16) = 3x^2(x - 4) - 4(x - 4) = (3x^2 - 4)(x - 4)$

The Difference of Two Squares

$a^2 - b^2 = (a + b)(a - b)$

$x^2 - 1 = (x + 1)(x - 1)$

$16x^2 - 1 = (4x + 1)(4x - 1)$

Simple Factoring

There is nothing in front of x^2:

$x^2 + 5x + 6 = (x + 3)(x + 2)$

ac Method

There is a number in front of x^2. Find two numbers that multiply to ac and add to b. Use those numbers to split up the middle term:

$6x^2 - 7x - 5 = 6x^2 + 3x - 10x - 5 = (6x^2 + 3x) - (10x + 5) = 3x(2x + 1) - 5(2x + 1) = (3x - 5)(2x + 1)$.

Perfect Squares

$a^2 + 2ab + b^2 = (a + b)^2 = (a + b)(a + b)$

$x^2 + 10x + 25 = (x + 5)(x + 5)$

$9x^2 + 24x + 16 = (3x + 4)(3x + 4)$

Chapter 17 Quiz

Question 1

$x^3 + 10x^2 + 16x = 0$

Find all the values of x by factoring this expression. x = _____ or x = _____ or x = _____

Question 2

Simplify:

$$\frac{x^2-9}{x+3}$$

Question 3

It is now some time in the future. All Americans in uniform including boy scouts are away fighting some war in a country most people wouldn't be able to locate on a map, and a foreign enemy has taken advantage of this situation by occupying the U.S. Enemy soldiers patrol the streets, schools are closed, and there are strict curfews in effect. Ordinary citizens are America's only hope. One morning you receive a letter in the mail that reads:

Dear parent,

Due to school closures, your son/daughter's algebra homework is being delivered by mail. Please have your student complete this work as soon as possible.

$x^2 + 21x + 20$
$20 + 21x + x^2$
$x^2 + 14x + 33$
$160 + 28x + x^2$
$x^2 + 10x + 25$
$70 + 19x + x^2$
$x^2 + 38x + 325$
$x^2 + 21x + 20$
$x^2 + 21x + 90$

$378 + 39x + x^2$
$x^2 + 14x + 13$

You immediately suspect that this is not a school assignment at all, because a) There are no actual equations here that seem to call for any kind of solution, b) some of the expressions are ordered backwards, and c) you don't have any kids. You know what to do immediately. Since you cannot find specific values of x, you discard the x's from your results. You end up with 12 pairs of positive numbers that have to be converted to the 26 letters of the alphabet. Thanks to the fact that the enemy has no idea what is going on you know that the conversion system is probably fairly simple. Your country depends on you and your algebra skills. Decipher the secret message.

____

Question 4

Find the solution to this system of equations:

$y = x^2 + x - 6$

$y = 2x + 6$

Find all the values of x.

x = _____ or x = _____

This problem should remind you of what you learned in chapter 5. Use the same principles to solve it. Notice that the first equation represents a curve and the second equation represents a line. The two solutions for x represent the x coordinates of the two points at which the line and the curve intersect.

Question 5

$2y = 4x + 6$

$y = x^2 + 2x - 1$

Find the two points where the line and the curve intersect. Use a graphing program to check your answer.

(_____ , _____) and (_____ , _____)

Portfolio Chapter 17

Create a quadratic expression by multiplying two binomials, like $2x - 1$ and $x + 5$. At least one of these binomials should have a number in front of the x, so that the resulting quadratic has a number in front of the x^2 term. Show how you factor the quadratic expression back to the original binomials.

Mommy, where do quadratic equations come from?

Yes, where do those quadratic equations come from? Well, we all know that milk comes from the store, and that quadratic equations come from math books. But it wasn't always that way. Way back in the old days, many thousands of years ago, when every little kid knew that milk comes from cows, quadratic equations came from squares and rectangles. Quadratic equations are actually named after these four–sided figures, since the root word "quad" means four. Long ago people considered such problems as: I need a barn with an area of 1800 square feet [or whatever unit they used in those days], and if I want the length twice as long as the width, how long would each side have to be? A mathematician would then be called in. The mathematician would immediately realize that the barn would have to be x feet wide and 2x feet long. To get the area, he would simply multiply the sides, so $x \cdot 2x = 1800$. This gives $2x^2 = 1800$, or $x^2 = 900$, which means that x would have to be 30 feet.

Mathematicians also looked at problems like: If the area of a rectangle is 10 square yards, and the length is 3 yards longer than the width, what is the width? We know that they did this kind of thing because we have actually found really old records of it. Anyway, if we call the width x, the length would be $x + 3$. We get the area, which is 10, by multiplying the sides, so $x(x + 3) = 10$ That multiplies to $x^2 + 3x = 10$, which rearranges to: $x^2 + 3x - 10 = 0$ Does that look familiar?

Now, if you're a good algebra student, you'll immediately grab a piece of paper and begin factoring this quadratic equation. You get $(x + 5)(x - 2) = 0$. This means that $x + 5 = 0$ or

$x - 2 = 0$. If $x - 2 = 0$, then $x = 2$. That's nice. The width of the rectangle is 2 yards, and the length is $x + 3$, which makes it 5 yards. That gives a nice rectangle with sides 2 and 5, and an area of 10. But what about the other solution, where $x + 5 = 0$? That means that $x = -5$. Now we all know that the width of a rectangle could not be -5. And the length in that case would have to be $x + 3$ or $-5 + 3$ which is -2. Hmmm.... There seems to be sort of a ghost rectangle sitting at 90 degrees to our real rectangle. It has a width of -5 and a length of -2, but the area is still 10.

Looking back at the first example, we notice a problem too. If $x^2 = 900$ we need to take the square root of both sides to find the answer. And 900 has two square roots, 30 and -30.

That is disturbing. And as we all know, we should not talk about disturbing things because it could cause mass panic in the general population. To avoid further problems of this type, mathematicians are no longer permitted to play around with rectangles, and algebra and geometry are kept strictly separated.

As for where quadratic equations really come from, just don't mention it in public.

Do You Believe in Ghosts?

Just in case you were personally disturbed by the previous resource, or even just curious, you should have a closer look at those ghost rectangles with the negative sides.

Consider a rectangle with a width of 5 inches, a length of 10 inches, and an area of 50 square inches. Call the width of this rectangle x. Calculate the value of x in two different ways. First, consider the length of the rectangle to be $x + 5$. Find the dimensions of the "ghost" rectangle with the negative sides. Next, consider the length of the rectangle to be $2x$, which also makes it 10 inches. Again calculate the width and length of the negative ghost rectangle. Hmm, that ghost is hard to pin down because it keeps shifting its position.

Do you think the ghost is real, or is it just a figment of your calculation?

Chapter 18: The Quadratic Formula

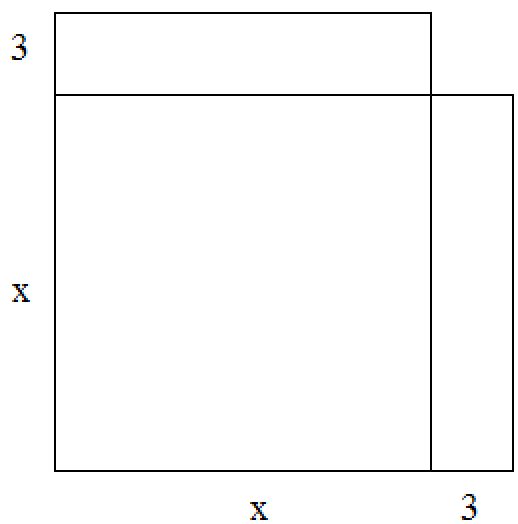
Completing the Square

Wading ever deeper into the swamp of quadratic equations... Did I say swamp? I meant, uh.., wetlands conservation area. Anyway, we are about to encounter a creature so fearsome that the very sight of it sends unsuspecting students running for their lives. This so–called swamp monster is the quadratic formula, and it is very impressive looking indeed. However, what most people don't realize is that this harmless creature feeds strictly on quadratic equations, most of which are completely indigestible to humans. Because it can attack any quadratic, it plays a vital role in the ecology by keeping the swamp from getting overgrown with unsolved equations. By carefully studying its habits, you can learn to get along with it without too much trouble.

To really understand the quadratic formula, we need to look at how other creatures hunt their prey. Take for example, a lizard hunting a worm. When the lizard is young and inexperienced, it may strike at the back end of the worm and struggle with it for a long time. When it is older and smarter, it will watch its prey carefully to determine which end is the head. Then it strikes quickly and efficiently to kill and consume the worm. In the same way, a wolf hunting a deer will know where its prey is vulnerable and go for the jugular, even though the bloodvessels are not visible through the skin. A simple quadratic equation like $x^2 + 6x + 8 = 0$ may not appear to us to have a vulnerable area. However, the quadratic formula looks at this equation and sees it as $x^2 + 6x + 9 = 9 - 8$. The original number at the end of the quadratic, 8 has been subtracted from both sides of the equation. A new and more favorable number, 9, has been added to both sides. Now notice that the middle number of the quadratic, 6, can be created by adding 3 + 3, and the 9 is just 3 times 3. With one bite of its razor–sharp teeth, the quadratic formula tears into its prey. The equation is ripped apart into $(x + 3)(x + 3) = 1$. Using its large square–root appendage to take the square root on both sides, our swamp monster breaks off both a positive and a negative square root, like this: $x + 3 = \sqrt{1}$, and $x + 3 = -\sqrt{1}$. As these smaller equations are chewed up they turn into $x + 3 = 1$ and $x + 3 = -1$. Digestion then produces the answers: $x = -2$ and $x = -4$. Put them back into the equation to make sure they are indeed correct. And from your biology classes you know what those answers are good for – they fertilize the swamp .

The process of seeing the equation in a different way so that it can be factored is also called "completing the square". Long ago, this method featured an actual square to be completed. The purpose of doing so was to find the area of the square, which would tell you the length of the sides, and therefore the value of the unknown x. It looked like this:

Suppose that you have an equation like $x^2 + 6x = 91$. You draw a corresponding picture that has a square with area x^2 and two rectangles with an area of 3x each.



From the original problem, you know that you have just drawn a figure with a total area of 91. To complete the square, you have to add a little piece with an area of 3 times 3, or 9. Once you have done that, your figure will have a total area of 91 + 9, or 100. Now that we know the area of the completed square, the solution is obvious just by looking at the picture. The only value for x that would give a completed square with a total area of 100 is x = 7. For cases where it is not immediately obvious, write out what you did:

$x^2 + 6x = 91$

$x^2 + 3x + 3x = 91$

$x^2 + 3x + 3x + 9 = 91 + 9$

$x^2 + 6x + 9 = 100$

The sides of the completed square are x + 3 long, so its area is $(x + 3)^2$

$(x + 3)^2 = 100$

$(x + 3) = 10$

$x = 7$

Because we are looking at a real square, we cannot consider the negative square root, −10, which would lead us to a value for x of −13. Thousands of years ago people either ignored these negative values or considered them evil. Today we know we can work with them, so the old simple method of completing an actual square has fallen out of favor, but the idea is still useful.

Let's do another example:

$x^2 + 10x + 2 = 0$. This is the same as $x^2 + 5x + 5x + 2 = 0$, or $x^2 + 5x + 5x = -2$

To complete the square, you need to add a little piece of 5 times 5, or 25, to both sides of the equation.

$x^2 + 5x + 5x + 25 = -2 + 25$

This is easily changed to $x^2 + 10x + 25 = 23$.

Notice that we can now factor the left side:

$(x + 5)(x + 5) = 23$

The square root of the left side is x + 5. To take the square root of the right side we have to remember that there are always two such roots: a negative square root and a positive square root. The ± symbol is used to indicate "plus or minus".

Therefore $x + 5 = \pm\sqrt{23}$

Or to put it another way: $x = -5 + \sqrt{23}$ or $x = -5 - \sqrt{23}$.

Practice completing the square on the following equations:

1. $x^2 + 2x + …. = 8 + ….$

2. $x^2 + 10x + …. = 11 + ….$

3. $x^2 + 20x + …. = 3 + ….$

4. $x^2 + 3x + …. = 0 + ….$ [Use a fraction rather than a decimal]

Answers:

1. $x^2 + 2x + 1 = 8 + 1$

   $(x + 1)^2 = 9$

   $x + 1 = 3$ or $x + 1 = -3$

   $x = 2$ or $x = -4$

2. $x^2 + 10x + 25 = 11 + 25$

   $(x + 5)^2 = 36$

   $x + 5 = 6$ or $x + 5 = -6$

   $x = 1$ or $x = -11$

3. $x^2 + 20x + 100 = 3 + 100$

   $(x + 10)^2 = 103$

   $x + 10 = \sqrt{103}$ or $x + 10 = -\sqrt{103}$

   $x = -10 + \sqrt{103}$ or $x = -10 - \sqrt{103}$

4. $x^2 + 3x + \left(\frac{3}{2}\right)^2 = 0 + \left(\frac{3}{2}\right)^2$

   $x^2 + 3x + \frac{9}{4} = 0 + \frac{9}{4}$

   $\left(x + \frac{3}{2}\right)^2 = \frac{9}{4}$

   $x + \frac{3}{2} = \sqrt{\frac{9}{4}}$ or $x + \frac{3}{2} = -\sqrt{\frac{9}{4}}$

   $x = -\frac{3}{2} + \frac{3}{2}$ or $x = -\frac{3}{2} - \frac{3}{2}$

   $x = 0$ or $x = -\frac{6}{2} = -3$

Notice that this last problem could be solved much faster by factoring:

$x^2 + 3x = 0$

$x(x + 3) = 0$

$x = 0$ or $x = -3$

Once you have completed the square a few times, you can see that it always involves dividing the number in front of x by 2. Suppose we call that number b:

$x^2 + bx + \ldots. = 0 + \ldots.$

To complete the square in a general way, we divide the number b by 2 and then square it:

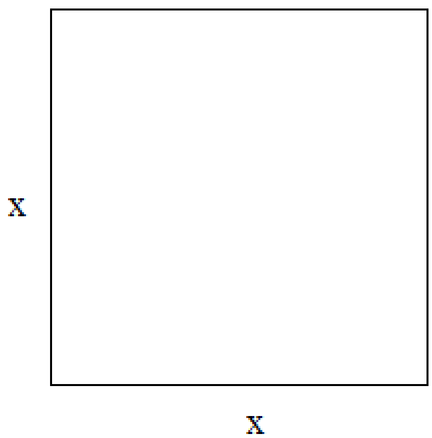
$x^2 + bx + \left(\frac{b}{2}\right)^2 = 0 + \left(\frac{b}{2}\right)^2$

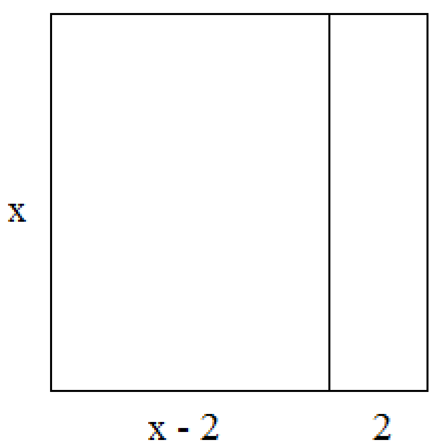
$\left(\frac{b}{2}\right)^2 = \frac{b}{2} \cdot \frac{b}{2} = \frac{b^2}{4}$

The general expression for a completed square is $x^2 + bx + c$. Here b and c are constants that replace the numbers you normally see in a quadratic expression or equation.

If you are asked to find c, it will be $\frac{b^2}{4}$. Go back to the problems above and use the formula $c = \frac{b^2}{4}$ to solve them. That is fast and it works, but in a few weeks you won't remember it. Drawing pictures and understanding what you are doing is better because it stays with you longer.

Now that you've had a chance to practice with positive numbers, we'll look at a quadratic equation with a negative number. Surprisingly, we can still use the ancient method of literally completing the square. Let's tackle $x^2 - 4x = 21$. First, we'll draw a square with sides x. Then we will attempt to subtract 4x by removing 2x from both sides:


x
x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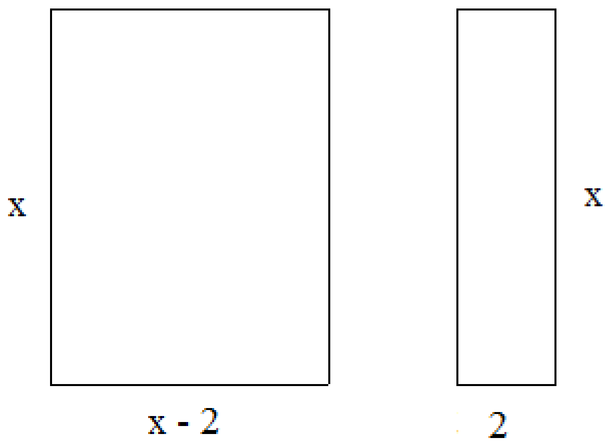
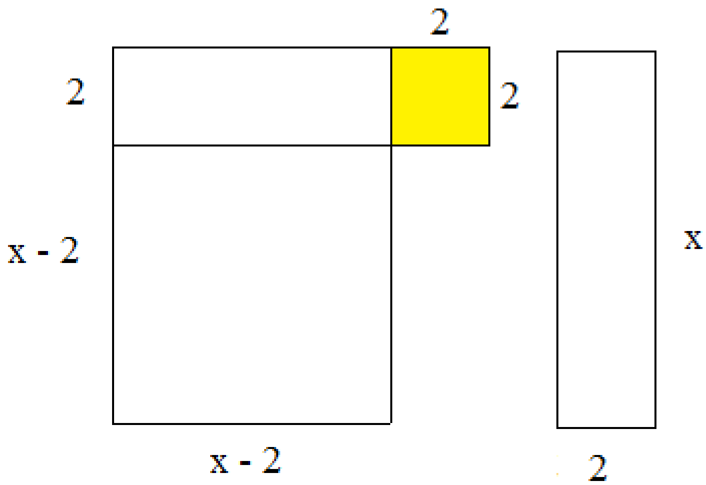
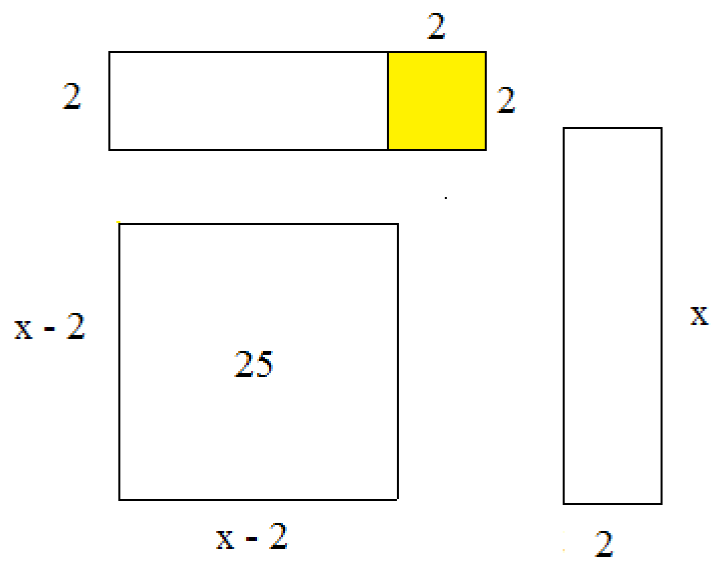


Now there is a problem, since there is not another piece with area 2x to remove. Let's fix that by adding a little piece. Just remember to add it on both sides of the equation.

$x^2 - 4x + 4 = 21 + 4$





$x^2 - 4x + 4 = 21 + 4$ can be factored into $(x-2)(x-2) = 25$, just as the picture shows. That is $(x - 2)^2 = 25$. Taking the square root on both sides, we get $x - 2 = 5$ or $x - 2 = -5$, giving values of 7 or −3 for x. Both answers are correct, but when you are using an actual square you would only consider x = 7 since the length of x cannot be negative.

Just like with factoring, you may be faced with an expression rather than an equation. To complete the square if there is no equals sign, try this trick:

$x^2 + 5x + 4$

To complete the square, we have to split the middle term into $\frac{5}{2}x$ and $\frac{5}{2}x$, which means that the little square to be added will have an area of $\frac{5}{2} \cdot \frac{5}{2}$, which is $\frac{25}{4}$. But now we can't add that on both sides! Well, not to worry, just add it and then subtract it:

$$x^2 + 5x + \frac{25}{4} - \frac{25}{4} + 4$$

$$(x + \frac{5}{2})(x + \frac{5}{2}) - \frac{25}{4} + 4$$

$(x + \frac{5}{2})(x + \frac{5}{2}) - \frac{25}{4} + \frac{16}{4}$

$\left(x + \frac{5}{2}\right)^2 - \frac{9}{4}$

Now go to http://www.shodor.org/interactivate/activities/AlgebraQuiz/. Select "no time limit" "level 1" and "quadratic". Rewrite the problem in a convenient form, and then complete the square as needed to find the value of x.

Completing Harder Squares

As you saw in the factoring chapter, not every quadratic equation is polite enough to start with x^2. How can we complete the square for something that looks like $4x^2 + 8x = 5$? Well, that 4 just has to move out of the way. If you have an equation, dividing both sides by the offending number works well. Otherwise, you have to temporarily stash it outside some parentheses.

$5x^2 + 10x = 6$

$x^2 + 2x = \frac{6}{5}$

$x^2 + 2x + 1 = \frac{6}{5} + 1$ Change 1 to $\frac{5}{5}$ to add it:

$(x + 1)^2 = \frac{11}{5}$

$x + 1 = \sqrt{\frac{11}{5}}$ or $x + 1 = -\sqrt{\frac{11}{5}}$, so $x = -1 + \sqrt{\frac{11}{5}}$ or $x = -1 - \sqrt{\frac{11}{5}}$

As we said in the chapter on Roots, math teachers don't like fractions under a square root sign. You should fix that before you hand in your work. Recall, or look back, to see that $\sqrt{\frac{11}{5}} = \frac{\sqrt{11}}{\sqrt{5}}$, and we can remove the square root on the bottom by multiplying the top and bottom by $\sqrt{5}$:

$$\frac{\sqrt{11}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{11}\cdot\sqrt{5}}{\sqrt{5}\cdot\sqrt{5}} = \frac{\sqrt{55}}{5}$$

This was actually a fairly easy problem to solve. As you are dividing by the first number and then dividing by 2 you usually end up with a lot of nasty fractions. Here a formula isn't a bad idea, so let's complete the square for the generic expression $ax^2 + bx + c$, where the point is to find c. First we have to move a out of the way:

$$a(x^2 + \frac{b}{a}x + \frac{c}{a})$$

To find $\frac{c}{a}$, we have to take $\frac{b}{a}$, divide it by 2, and then square it. Since $\frac{b}{a}$ is a fraction, division by $\frac{2}{1}$ is accomplished by multiplying by $\frac{1}{2}$. $\frac{b}{a} \cdot \frac{1}{2} = \frac{b}{2a}$. [Notice that you could divide by 2 quickly if you just slide the 2 underneath the division line]. Then square $\frac{b}{2a}$:

$$\frac{b}{2a} \cdot \frac{b}{2a} = \frac{b^2}{4a^2}$$

This tells us that $\frac{c}{a} = \frac{b^2}{4a^2}$. That's not quite what we want, because we are looking for a shortcut to find c. To get c, multiply both sides by a:

$$a \cdot \frac{c}{a} = \frac{b^2}{4a^2} \cdot a$$

$$\frac{ac}{a} = \frac{ab^2}{4a^2}$$

$$c = \frac{b^2}{4a}$$

Okay, let's see if that works. Going back to $5x^2 + 10x = 6$, we have a = 5 and b = 10. To find the missing c that will complete the square, we will use $c = \frac{b^2}{4a}$:

$$c = \frac{10^2}{4 \cdot 5}$$

$$c = \frac{100}{20} = 5$$

Add 5 to both sides: $5x^2 + 10x + 5 = 6 + 5$

$5x^2 + 10x + 5 = 11$

So that is a perfect square on the left of the equation. But wait, what is the square?? The first number, 5, doesn't make a nice square at all. To remedy this situation, we have to change that to something more convenient, like maybe 25. Fortunately we can accomplish this by multiplying both sides by 5 to get $25x^2 + 50x + 25 = 55$. Now it is possible to make a square on the left:

$(5x + 5)^2 = 55$

$5x + 5 = \sqrt{55}$ or $5x + 5 = -\sqrt{55}$

$5x = -5 + \sqrt{55}$ or $5x = -5 - \sqrt{55}$

$x = -1 + \frac{\sqrt{55}}{5}$ or $x = -1 - \frac{\sqrt{55}}{5}$

After a lot of practice solving quadratic equations, you'll want even more of a shortcut. Here is where the quadratic formula comes in. Instead of a quadratic equation with specific numbers, consider a generic one like $ax^2 + bx + c = 0$. That does look a little scary, so let's go with a=1 to get $x^2 + bx + c = 0$. Just like you did with all the other equations, you want to factor this one into $\left(x+\frac{b}{2}\right)\left(x+\frac{b}{2}\right)$ = ... In order for you to be able to do that, you have to complete the square by adding $\left(\frac{b}{2}\right)^2$ or $\frac{b^2}{4}$ to both sides of the equation. Starting with $x^2 + bx + c = 0$ and moving the c to the other side, we get $x^2 + bx = -c$. Adding $\left(\frac{b}{2}\right)^2$ to both sides gives us $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(\frac{b}{2}\right)^2 - c$.

Now we can factor this into $\left(x+\frac{b}{2}\right)\left(x+\frac{b}{2}\right) = \frac{b^2}{4} - c$.

When taking the square root of this on both sides, don't forget the negative square root:

$\left(x+\frac{b}{2}\right) = \sqrt{\frac{b^2}{4} - c}$ **and** $\left(x+\frac{b}{2}\right) = -\sqrt{\frac{b^2}{4} - c}$.

We can write both square roots into one expression using the ± sign:

$\left(x+\frac{b}{2}\right) = \pm\sqrt{\frac{b^2}{4} - c}$. This expression gives both answers for x:

$x = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - c}$.

The unwieldy expression $-\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - c}$ can be cleaned up a bit:

$\frac{b^2}{4} - c$ can be written as $\frac{b^2}{4} - \frac{4c}{4}$, which can be changed to $\frac{b^2-4c}{4}$. The reason for doing that is that $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$, so $\sqrt{\frac{b^2}{4} - c} = \frac{\sqrt{b^2-4c}}{\sqrt{4}} = \frac{\sqrt{b^2-4c}}{2}$. This means that we can change

$-\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - c}$ to $-\frac{b}{2} \pm \frac{\sqrt{b^2-4c}}{2}$.

Now our two fractions have the same denominator, so we change our expression into $x = \frac{-b \pm \sqrt{b^2-4c}}{2}$. This is a baby quadratic formula. Isn't it cute? At least, I hope you think it's cute. This particular baby is an orphan, and it will need help and care in order to grow up. Some schools have a course where you take care of a computerized baby doll and carry it with you all day. This is much more fun. Take your baby swamp monster and feed it the easy quadratic equations at http://www.shodor.org/interactivate/activities/AlgebraQuiz/. (Rewrite each equation so that it looks like $x^2 + bx + c = 0$, then plug the values for b and c into your formula). When it can handle quadratic equations that have $a > 1$ (medium and hard), it will be all grown up. Full grown quadratic formulas usually handle these equations by dividing both sides by a. Just so you know what your cute little formula will look like when it is grown up, here is a picture: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. It will take all of your algebra skills to help your baby get there [see Assignment Chapter 18]. Once quadratic formulas become imprinted on a human, they usually stay with you for a long time. When you are done with school, you may want to release your quadratic formula back into the wild....

Quadratic Equations and Death Rays

Death rays?? I didn't know this stuff could be dangerous. We'd better have a closer look at this. Let's graph a quadratic equation. Most of the time you'll want to assign this job to a computer or graphing calculator, but for now we're trying to learn something. No matter how often a computer draws a graph it never really seems to get anything out of it, so you'll need to do your own graphs at first. When we graphed simple equations involving y and x, we called them linear equations because they looked like a straight line. Once you start squaring x, something else happens: y begins to increase faster and faster.

Take out your graph paper, and graph $y = x^2$. Just put down dots for now, we'll connect them later.

You'll notice that y gets bigger very quickly and your graph will run off your paper long before x even gets to 10. Also graph for negative values of x. For each negative value of x, the y value is the same as for the positive x value.

Now connect the dots with a smooth curve. The resulting shape is called a **parabola**. You may have heard this word before. Parabolic mirrors have a special property: they are able to focus sunlight that falls on them to a single point. Here is a movie showing how powerful a parabolic mirror can be: http://www.youtube.com/watch?v=0xfTl3Ugyjk&feature=related

The mirror in this video looks nothing like the parabola on your paper, but parabolas can be manipulated mathematically. Try graphing $y = \frac{1}{10}x^2$. You should see a much flatter parabola.

The mathematician Archimedes is said to have used this principle in 212 BC to burn a fleet of ships. Read about this amazing legend and a modern–day test at http://web.mit.edu/2.009/www/experiments/deathray/10_ArchimedesResult.html

All quadratic equations look like parabolas when they are graphed. That may not seem obvious at first, but it becomes clearer when you use the principle of completing the square. For example, take the equation $y = x^2 + 6x + 6$. We can complete the square like this:

$y + 9 = x^2 + 6x + 9 + 6$

$y + 9 = (x + 3)^2 + 6$

$y = (x + 3)^2 - 3$

When you graph this equation, it will look just like your first parabola, $y = x^2$, except that it will be 3 units further down (below the x–axis) and 3 units to the left.

By completing the square, you can take any quadratic equation and change it to resemble the basic function $y = x^2$, which can also be written as $f(x) = x^2$ [see "Functions"]. Any quadratic can be changed like this, and once we do that it becomes much easier to predict what the graph will look like. Let's try a tough one: $f(x) = 4x^2 + 24x - 3$. When you go to change a function, it is easier to use a different notation. Recall that f(x) is just y:

$y = 4x^2 + 24x - 3$

Now we want to complete the square, but our method for that didn't include having a number like 4 in front of x^2. To get rid of "multiply by 4", we need to divide by 4, carefully so that we don't change the equation to a different one. Just divide everything by 4:

$\frac{y}{4} = x^2 + 6x - \frac{3}{4}$

To complete the square for $x^2 + 6x$, we have to add 9 on both sides of the equation:

$\frac{y}{4} + 9 = x^2 + 6x + 9 - \frac{3}{4}$

$$\frac{y}{4} + 9 = (x + 3)^2 - \frac{3}{4}$$

$$\frac{y}{4} = (x + 3)^2 - \frac{3}{4} - 9$$

$$\frac{y}{4} = (x + 3)^2 - \frac{3}{4} - \frac{36}{4}$$

$$\frac{y}{4} = (x + 3)^2 - \frac{39}{4}$$

That completes the square, and we make it look like a function again by multiplying everything by 4:

$y = 4(x + 3)^2 - 39$

$f(x) = 4(x + 3)^2 - 39$

This is a quadratic function, and its graph is a parabola.

You'll be allowed to learn more about parabolas after we've monitored your activities for a while to make sure you won't be doing anything evil with them.

The Vertex of a Parabola

Parabolas are always symmetrical. The line of symmetry passes through the vertex, which is the "tip" of the parabola. If the parabola opens upwards the vertex is the lowest point, and if the parabola is upside down the vertex is the highest point. You can graph a parabola in a coordinate system if you write its equation as $y = ax^2 + bx + c$. Just plug in a value for x, and out comes a value for y. The easiest value to pick for x is 0. For example, if $y = 2x^2 + 4x + 1$, then when $x = 0$ y will be 1. Usually you can see the shape of the parabola just by picking a few additional small positive and negative values for x. Although graphing programs provide you with a convenient way to draw parabolas, you should also make sure you can make a decent sketch with pencil and paper.

If you play around with parabolas for a bit, you will see that parabolas open upward when a, the number in front of x^2, is positive. A negative value for a will cause the parabola to open downward so it looks upside down. When a is large the parabola will be narrow, and smaller values for a result in a wider, less steeply curved parabola.

When the graph crosses the x–axis, y is equal to 0. Many parabolas are entirely above or below the x–axis, so they never cross it. Others have two points where they cross the axis, and some just touch the x–axis at their vertex. Points where a parabola touches or crosses the x–axis are called the zeroes. Because parabolas are symmetrical, the vertex is always located on the axis of symmetry, and the axis of symmetry runs exactly between the zeros. We would therefore expect that the x–coordinate of the vertex would lie in the middle between the zeroes, or on the zero if there is only one. Parabolas that don't have zeros would have them if they were shifted down or up, and the line of symmetry still runs between where those zeroes would be. You will learn more about shifting parabolas in algebra II.

If you know where the zeroes are, or if the equation is simple enough that you can find them easily by factoring, you can find the x–coordinate of the vertex by taking the average of the value of the x–coordinates. For example, for the quadratic function $y = 6x^2 + 30x$, the zeroes are easy to locate. Set y equal to zero, and factor: $0 = 6x^2 + 30x$, so $6x(x + 5) = 0$. Either 6x is 0, in which case x = 0, or x + 5 = 0 so that x = –5. Add 0 and –5, and divide by 2. The x–coordinate of the vertex is 2.5.

For more complex quadratics the zeroes can be found by using the quadratic formula to solve the equation $y = ax^2 + bx + c$ when y = 0.

For $0 = ax^2 + bx + c$, the quadratic formula tells us that $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. If you know how fractions work, you can also write this formula as $x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$ The ± sign means that the formula provides two solutions, unless $\sqrt{b^2 - 4ac}$ happens to be equal to 0. Those solutions are the zeroes, and they are located at equal distances from the axis of symmetry. That means that $x = -\frac{b}{2a}$ is the axis of symmetry. The vertex always lies on the axis of symmetry, so its x–coordinate must be $-\frac{b}{2a}$. To find the y–coordinate of the vertex, just plug the x–coordinate into the equation that describes the parabola.

Assignment Chapter 18

Solve $ax^2 + bx + c = 0$ for x. Show all the steps to get to $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Chapter 19: Choices and Chances

Life is full of choices, but you don't really notice how boring and irrelevant most of them are until you read an algebra book on the subject. Math problems about choices often involve persons with amazingly limited wardrobes making choices about what to wear with complete disregard for what clothes go together. A sample problem would look like this: "John has three pairs of pants and 4 shirts. How many different outfits can he wear?" Since we don't want you learning how to dress like a mathematician we'll do something else instead. Let's visit Pete's 101 Flavor Ice Cream Palace.

Pete's Palace has 101 actual flavors of ice cream, ranging from "Almond Crunch" to "Tangy Tomato". If you want a single–scoop cone, you have exactly 101 choices. However, as every ice cream fan knows, the cone itself is very important too. Pete offers regular cones, sugar cones, and waffle cones. Now how many choices do you have? Well, if you pick a regular cone, you have 101 choices of ice cream flavors. If you pick a sugar cone, you again have 101 choices of flavor. Then if you pick a waffle cone, there are another 101 choices. So the total number of choices is 3 times 101, or 303 different choices that you can make.

My personal choices are more limited, because I always choose a sugar cone. Now, with that many flavors to choose from, how can anyone just have one scoop? Imagine that you have already chosen your favorite cone, and now you want two different flavors of ice cream. How many choices do you have? Once you have chosen your first flavor out of 101 choices, you have 100 flavors left for your second choice. This may lead you to believe that you have 101 times 100 choices, because for each of 101 flavors you can pick from 100 second scoops. Since all of this is entirely imaginary, and I can certainly imagine having an unlimited amount of money, let's ask Pete to actually create all of these different two–scoop cones. Pete, being entirely imaginary also, happily obliges. He even has a very large rack that he sets on the top of his counter to hold all the cones. Finally the job is all done. As you look over all 10,100 cones, deciding which one to pick, you notice a problem. Some of the cones have the same two flavors. You spot one cone that has "Blueberry" on the bottom, and "Cheese Pizza" on the top, and another cone that has "Cheese Pizza" on the bottom and "Blueberry" on top. In fact, for every combination there is a second cone with the flavors reversed. For people who are not ice cream connoisseurs, the choices have just been cut in half, to only 5,050. But if you truly appreciate ice cream flavors, you know that the order really does matter. If you have "Banana" and "Black Licorice", the whole cone is just ruined if the licorice is on top. Licorice is a strong flavor, and it keeps the palate from appreciating the more delicate banana afterwards. Therefore, if the order matters to you, you have 10,100 choices for your double–scoop cone.

Mathematicians call these selections where the order matters **permutations**.
Being picky like that results in more choices because none of your choices are the same. When you just want to select some things and you don't care about the order, that is called **combinations**. For combinations you always end up with fewer choices because some of the choices you make are actually the same. I read online somewhere that the best way to remember these names is to call all combination locks "permutation locks", since the order of the numbers is definitely important there.

For those people who just can't decide on only two flavors, there is of course the triple–scoop cone. This is served only in waffle cones because they are larger. Assuming the order of the flavors matters to you, how many choices (permutations) do you have now? If you want your flavors to all be different, you have 101 choices for your first scoop, which goes on the bottom, 100 choices for the middle scoop, and 99 choices for the last scoop, for a total of 101 times 100 times 99 = 999,900 choices. Wow, that's a lot of cones!

Pete also offers "triple scoops in a dish". Putting the scoops in a round dish avoids the problem of which flavor to put in first since the dish is large enough to put the flavors side by side. This means that we are looking at combinations, and there are fewer choices. "Caramel Apple" – "Strawberry" – "Root Beer" will end up the same as "Strawberry" – "Caramel Apple" – "Root Beer" or " Root Beer " – "Caramel Apple" – "Strawberry" or.... Exactly how many different ways are there to arrange these three flavors? My first thought is to say, "I don't care, let me eat my ice cream now", but we all have to make sacrifices for math. Sigh... Mathematicians always handle problems like this in a logical, organized way, so let's try to copy that. Here are the possibilities:

Caramel Apple – Strawberry – Root Beer
Caramel Apple – Root Beer – Strawberry

Strawberry – Root Beer – Caramel Apple

Strawberry – Caramel Apple – Root Beer

Root Beer – Caramel Apple – Strawberry

Root Beer – Strawberry – Caramel Apple

Notice the pattern here. When we pick Caramel Apple as the first flavor, there are two choices for the order of the other two flavors. The same thing happens when we pick Strawberry first, or Root Beer. This happens because there are 3 choices for the first flavor, followed by two choices for the second flavor, followed by only one choice for the third. There are 3 times 2 times 1, or 6 ways to arrange these flavors that all end up being the same.

Now, imagine Pete happily creating all those dishes of ice cream, with 101 choices for the first scoop, 100 choices for the second scoop, and 99 choices for the third scoop. He is wasting a lot of time now, because for each combination of three flavors, he is making 6 dishes that are the same. The total number of **different** dishes is 101 times 100 times 99, **divided by** 6.

That was "choices" in a nutshell, or rather in an ice cream dish. Now let's look at chances.

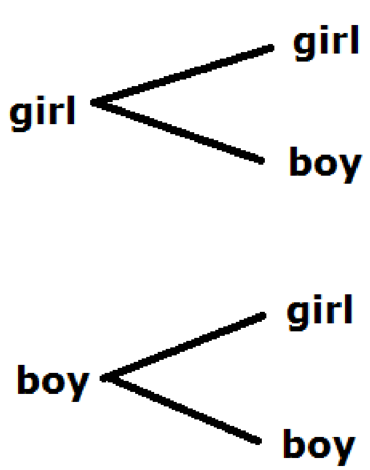
Mrs. Jones walks into Pete's Palace and orders a single–scoop cone. Mrs. Jones is a nice respectable middle–aged lady who occasionally worries about what the neighbors think of her. The chance that she will order "Chocolate/Chocolate–covered Ants" is considerably less than 1 out of 101. In fact, researchers have already determined that when faced with a choice of 101 flavors of ice cream, the average person will order...... vanilla. This is not the type of "chance" that we will be discussing, because we're going to look at absolutely random sorts of chances.

Flipping a coin is a random event. There are two choices for the outcome: heads or tails. The chance of one particular outcome, say heads, is 1 out of 2, or $\frac{1}{2}$. If you roll a die, there are 6 possible outcomes. The chance that you'll roll a four is 1 out of 6. The idea behind this is simple: determine the total number of possibilities, and then look only at the ones you want. For example, suppose that for some reason, you really want to get an even number when rolling a die. The total number of choices for the outcome of rolling a die is 6. Only 3 of those are even numbers. The chance of rolling an even number is 3 out of 6, which is the same as $\frac{1}{2}$, or 50%. Because there are some dice that have weights so you can cheat with them, you'll often see math problems specifying that you are "rolling a fair die". This is simply meant to indicate that the outcome will be entirely random so that you can apply the usual methods for calculating probability.

Unlike in real life, probability in high school math problems is usually quite straightforward. The trick is mostly to correctly determine the total number of outcomes, and then decide how many of those outcomes you are interested in.

Once you've been doing calculations with chances for a while, chances are you'll want to take a shortcut. To consider all of the available possibilities when you are only interested in one just starts to get a little tedious. Is there a shortcut?

The answer is yes, but you do have to be careful. Let's consider Mr. and Mrs. Smith, a nice average couple wanting to have two children. Like many couples they would like to have a boy and a girl. Although the probability of a boy being born is actually slightly greater than that of a girl being born, for practical purposes we'll say that both probabilities are equal at 1 out of 2, or 1/2. What are the chances that the Smiths will have 2 girls? Well, the probability of the first girl's birth is 1/2. Once that has happened, there is another 1/2 probability that there will be a second girl. The best way to see what is going on is by making a “tree” diagram:



The outcome we are looking for is in the top half of the tree diagram, and it is only half of that part. 1/2 of 1/2 is the same as 1/2 times 1/2, which is 1/4. One out of 4 is the actual probability of Mrs. Smith giving birth to two girls. To check that this is correct, look at all of the available outcomes:

1 girl + 1 girl
1 girl + 1 boy
1 boy + 1 girl
1 boy + 1 boy

Two girls being born is one of four possible outcomes.

Now what are the chances of the Smiths having a boy and a girl? You might want to say that it is also 1/2 times 1/2, but notice that the combination of a boy and a girl can occur in two separate ways. The girl could be born first, followed by a boy, or the boy could be born first. The probability of the first child being a girl is 1/2, and the probability of a girl followed by a boy is 1/2 times 1/2, or 1/4. In a separate outcome, the boy could be born first, followed by a girl. The probability of this event is also 1/2 times 1/2, or 1/4. Since Mr. and Mrs. Smith would be equally happy with either outcome, the probability of them getting the boy and the girl they want is 1/4 + 1/4 or 1/2. Again, check this by looking at the table: there are 4 possible outcomes and 2 of those are the ones Mr. and Mrs. Smith find desirable. 2/4 = 1/2.

All of this tells us that we can take a shortcut by multiplying probabilities for two (or more) events that are independent of each other, like babies being boys or girls, or coins coming up heads or tails. We just have to watch carefully to see if the desired outcome could occur in more ways than one, and if so we have to add the probabilities of each of these ways.

The probability of rolling a 4 when you roll a die is $\frac{1}{6}$. That means that the probability of NOT rolling a 4 is $\frac{5}{6}$. The total probability here adds up to 1, which is kind of a comforting thought. You can safely say, with 100% certainty that when you roll a die, you will either roll a 4, or not roll a 4. Although you probably already knew that, now you know it in a more complicated way.

Sometimes though the fact that the total probability adds up to 1, or 100%, can be useful. For example: You are visiting a tropical island for a week during the rainy season, and the probability of rain is 60% each day. What are the chances that it will rain on at least one day during your visit? [Been there, done that, don't recommend it. Although the actual water is warm what passes for rain over there is more like a simulation of what it would be like to stand under Niagara Falls]. The problem seems fairly complex. It could rain just one day of the visit, and there would be no rain on the other 6 days. There is a 40% chance of no rain, or .4. We could multiply the probabilities like this: 0.6 times 0.4 times 0.4 times 0.4 times 0.4 times 0.4 times 0.4. However, one day of rain could occur in 7 different ways so we'd have to multiply by 7. Then we'd consider that it could rain 2 days of the visit, and so on.

In this case you may want to take the easy way out by calculating the opposite probability: that it rains on no days during the visit. This somewhat unlikely event can occur in only one way: all days are the same with no rain. The probability of this one event is easy to calculate: 0.4 times 0.4 times 0.4 times 0.4 times 0.4 times 0.4 times 0.4, or 0.4^7, which is 0.0016. Therefore the probability of one or more days of rain is the opposite, or 1 – 0.0016. That works out to over 99%, so you're pretty much guaranteed to get soaked.

Next, let’s go through an example of a type of problem you might see on the SAT. Suppose a large firm has accepted 9 interns [trainees] for the summer. In the fall, five of these interns are offered a permanent job with the company. Samantha, one of the unsuccessful interns, comes to you and says that she thinks she did not get hired because the company prefers male employees. She points out that there were 5 male and 4 female interns, and permanent jobs were offered to 4 men and only 1 woman. Samantha wants you to calculate how likely it is that this outcome is due to chance alone.

To solve this problem, you must first consider all of the possibilities, and then look at the specific possibility that 4 men and 1 woman would be chosen. At first things may seem a bit confusing, but actually a random selection of 5 people out of a total of 9 would have nothing to do with gender. It is just a simple problem: in how many ways can you select 5 out of 9? Does the order matter? No, it does not. That means we are dealing with combinations, which restricts the total number of choices. First, let’s select all five people. There are 9 choices for the first person, 8 for the second, 7 for the third, 6 for the fourth, and 5 for the fifth, for a total of 9 x 8 x 7 x 6 x 5 choices. However, because the order doesn’t matter we have to consider how many of those choices would end up being the same. Just like there are 3 x 2 x 1 ways to arrange 3 scoops of ice cream, there are

5 x 4 x 3 x 2 x 1 ways to arrange 5 people (invent some names and try it out if you're not sure). The final number of choices is $\frac{9 \times 8 \times 7 \times 6 \times 5}{5 \times 4 \times 3 \times 2 \times 1}$. The 5 cancels out, and 4 x 2 on the bottom cancels with 8 on the top. There is also a 6 on top that can be divided by the 3 on the bottom to give 2. So, you only need to multiply 9 x 7 x 2, which is 126.

Now, let's consider how many of these choices involve selecting 4 men and 1 woman from 5 male and 4 female interns. Just how many different ways are there to do that? Well, I have 5 choices to pick the first man, 4 remaining choices for the next man, 3 for the third man, and 2 for the fourth. That is a total of 5 x 4 x 3 x 2. The order doesn't matter, so I have to divide that by 4 x 3 x 2 ways to arrange 4 people. Therefore there are only 5 different ways to select 4 men from a group of 5. You can also look at it the opposite way: there are 5 different ways to leave 1 man out. Next, I have to select 1 woman from among 4 different female interns. There are 4 ways to do that, for each of the 5 ways that I select the men. The total possibilities for that are 5 x 4 = 20.

The chance that 4 men and 1 woman would be selected is $\frac{20}{126}$ or about 16%.

When I made up this problem I thought that it might suggest that the company discriminated against women, but with more than a 1 in 7 chance that the outcome would occur randomly there is just not enough to back up such an allegation.

Let's Experiment with Probability

Read the web page and take the quiz:
http://www.mathgoodies.com/lessons/vol6/intro_probability.html

Improving Your Chances of Understanding Probability

Here is another explanation about basic probability, from the online book I'll Give You A Definite Maybe: http://records.viu.ca/~Johnstoi/maybe/maybe1.htm. Read up to and including "G. Self–Test on Simple Probabilities". It is not necessary to read the section on the Binomial Distribution, which is covered in statistics courses.

Probability on Math TV

Watch the probability problems on Math TV: http://www.mathplayground.com/mathtv.html

Try it on your own after seeing the teacher's explanation.

Chapter 20 Quiz

Question 1

A box of small candies with different colors is almost empty. There are only 12 candies left. 3 of those are yellow, and 9 of them are red. If you randomly shake a single candy from the box into your hand, what is the probability that it will be a yellow candy? Express your answer as a fraction in lowest terms.

____

Question 2

We saw earlier that if the order in which you make your choices is important, you end up with the maximum number of possibilities. In word problems, a situation where the order matters is often shown like this: The (insert really boring activity here) club has 10 members. In how many different ways can the club elect a president, a treasurer, and a secretary? Notice that you are being asked to make 3 choices. Each time you make a choice, there are fewer possibilities left for the next choice. What is the answer?

____________ different ways

Question 3

The classical question in which the order of choices does not matter involves sports matches of some kind, like soccer or tennis. It usually goes like this: If a soccer league has 6 teams, and each team needs to play all the other teams, how many games would there have to be? Again think of this in terms of choices. For each game, you choose two teams. However, team 3 playing against team 5 is the same as team 5 playing against team 3. So, how many games will have to be played?

__________ games

Question 4

On his way to work, Brian has to deal with only two traffic lights. Both lights are independently set to be red for half of their cycle, and green the other half of the time. What is the probability that Brian will encounter two red lights on his way to work?

a. 1/8

b. 1/4

c. 1/2

d. 1

e. The chances are much higher if Brian is late for work

Question 5

The combination of one red and one green light can occur in two different ways. What is the probability that as Brian drives to work, he will encounter one red traffic light and one green light?

a. 1/8

b. 1/4

c. 1/2

d. 3/4

e. 1

Question 6

When tossing a fair die, what is the probability of obtaining a 1 **or** a 2 **or** a 3? Express your answer as a decimal number.

[Note that the probability of obtaining 1 **or** 2 **or** 3 **or** 4 **or** 5 **or** 6 is equal to 1]

____

Question 7

Cassie's schedule has changed, and now she has to get up much earlier. She always has toast for breakfast, but due to being unusually tired she will drop her buttered toast on the floor three times this month. What are the chances that the toast will land with the buttered side down at least one of those times? Express your answer as a fraction.

____

[In spite of results to the contrary obtained on the TV show "MythBusters", it's just common knowledge that dropped toast is much more likely to land with the butter side down. This is actually caused by the Law of Entropy. This law states that the Universe tends towards maximum randomness. A buttery mess on the floor is a more random state of affairs than a piece of toast that can be picked up cleanly. For more information on this topic, see my new book: Is the Universe Conspiring Against Us?]

Question 8: Challenge Question

There are three cards in a box. The first card is white on both sides. The second is red on both sides. The third card has one side that is white and one side that is red. A card is selected from the box at random and placed on a table. The side that is visible is red. What are the chances that it is the card that is red on both sides?

Chapter 20: Statistics

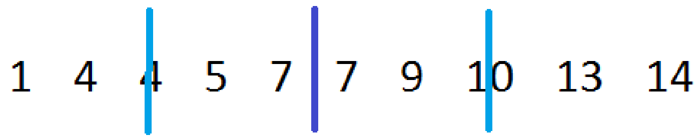
Experimental science involves observations, measurements, and counting. If you observe, measure or count for a long while, and write everything down, you end up with a collection of data. Often these data are numbers, so we call them numerical data or **quantitative** data; they represent a quantity (an amount). If the data are not numbers we call them **qualitative**, and we sort them into categories. Qualitative data are also called **categorical data**. There are many ways to organize data and compare them. Statistics is the science of dealing with data and making sense of them.

Important terms to know include **mean**, **median**, **range**, **quartiles**, and **interquartile range**. The **mean** is the average of the numbers that make up the data. Simply add up the numbers and then divide by how many numbers you have.

To find the **median**, order your numbers from smallest to largest. The median is the middle point of these numbers, which means that it is either the *middle number* or the *average of the two middle numbers*. For the data set 2, 4, 4, 7, 8, the median is 4 because that is the middle number. For 5, 6, 8, 200, the median is 7 because that is the average of 6 and 8.

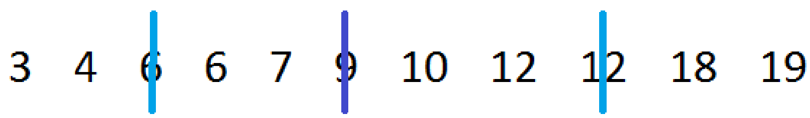
The median has a definite advantage over the mean if your data contains a few very high or very low values, because such values can really pull the average in one direction. If the *average* grade on a test is 60%, that could mean that most students did really well, but the teacher gave Joe and Maria zeros for cheating. If the *median* grade is 60%, that means that half the students scored at or below 60%, and half the students scored at or above 60%. The median is usually used for house prices, so that a few very expensive homes don’t make it seem like the average house is more expensive than it actually is.

The median divides the data in half, and when you divide those halves in half again you get the **quartiles**. This division is done the same way as for the median. If the median is between two numbers, these numbers are included in the half that you divide to get the quartiles:



In the picture above, there are an even number of data points. The median (Q2) is 7, the lower quartile (Q1) is 4 and the upper quartile (Q3) is 10.

If there are an odd number of data points, there is an actual middle number that is the median. Put a line through the middle number and then ignore it as you look for the quartiles:



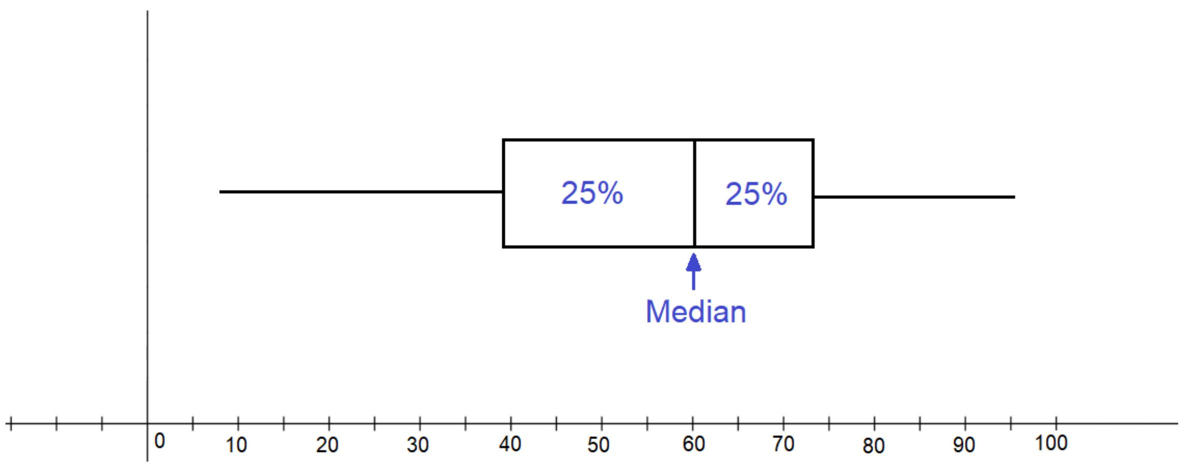
Just remember that 50% of the data are always at or below the median, and the other 50% are at or above the median. 25% of the data are located in each quartile.

The **interquartile range** is the distance between the upper and lower quartile. For the data set above, the interquartile range is 12 minus 6, which is 6.

Outliers are data points that don't seem to "fit" with the remaining data because their value is too large or too small. Outliers may represent an unusual event or an error in measurement. If you are asked to calculate if a particular point is an outlier, find the interquartile range and multiply it by 1.5. Then add this figure to the number that represents the top quartile and subtract it from the number that represents the bottom quartile. A value outside this range may be considered an outlier. For the data set above, 3, 4, 6, 6, 7, 9, 10, 12, 12, 18, 19, the upper quartile is at 12 and the lower quartile is at 6. The interquartile range is 12 – 6 = 6. 1.5 times the interquartile range is 6 x 1.5 = 9. There should be no numbers that are more than a distance of 9 away from the upper and lower quartiles. The lower quartile is 6, so there is no problem at the low end. The largest number in this data set is 19, which is only 7 more than the upper quartile. If the highest number was 25, it would be an outlier because 25 – 12 = 16, which is definitely more than 9. 12 + 9 = 21, so any number higher than 21 will be considered an outlier

The **range** is the difference between the highest value and the lowest value. It shows the "spread" of the data. If the range is large, that could be because there is just a single very low or very high value, or it could mean that all of the data points are very spread out.

A **boxplot** (box-and-whisker plot) shows the median, the upper and lower quartiles, and the range. The sides of the box represent the lower and upper quartiles, while the median is indicated by a line inside the box. The whiskers may extend to the highest and lowest values in the data set, or unusually high or low values may be marked separately with an asterisk. The box and its whiskers may be oriented horizontally or vertically. Boxplots provide a good visual representation. The boxplot below shows fictional grades from Mrs. Markham's 9th grade social studies classes:



The first thing that you might notice here is that the median line is not in the middle of the box, so how could both quartiles be 25%? Well, as it turns out the grades in the lower quartile are further apart than those in the upper quartile. The number of grades in each quartile is the same, but their range is not. Students in the lower quartile scored between 40 and 60 percent, but for students in the upper quartile the grades had a smaller range, between 60 and 74%. The left whisker also extends further than the right, but the facts are the same as always: 50% of the data points are below the median, and 50% are above. The odd-looking distribution of the actual values of the grades is caused by the fact that many students in Mrs. Markham's classes are recent immigrants who are at various stages of learning English.

If we want to know more about the variability of a set of data, we should look at each data point separately. How far is each point from the mean? Well, we could take each value and subtract the mean from it. For example, if a particular value is 14, and the mean is 9, then we would subtract: 14 – 9 = 5. If a value is below the mean, You get a negative number: 7 – 9 = -2. Because the mean is the central value, if you try to get the average of these numbers you will find that they add to zero. That is not useful. It is a lot smarter to use the absolute values. Now we are using the distance from the mean, and all of these values will be positive or zero (absolute values are never negative). So, for a value of 7 and a mean of 9, calculate like this: $| 7 - 9 | = 2$. Then find the mean (the average) of these absolute values, which is the **mean absolute deviation**. Be sure to include any zero values!

Using absolute values is not the only way to get rid of negative values so we don't get zero when we add them. Another way is to take the square, like this: $(14 - 9)^2 = 25$, and $(7 - 9)^2 = 4$. Notice that when you do this a large deviation from the mean like 5 is counts for more than a small deviation of -2. It actually makes sense to do that, because small deviations from the mean are more likely to occur than large ones. When you add all of the squares and average them, you get the **variance**. The **standard deviation** is the square root of the variance. The standard deviation is the most common way to measure variability in data.

Frequency Distribution

If your data can be neatly grouped into categories, you can present them in a **frequency table**, as shown in the next example.

Merriam Porter High School was having budget problems. Some elective subjects would have to be discontinued. In order to find out which classes were the most popular, administrators asked all of the students at the school to name their favorite subject. The responses were counted carefully, resulting in a set of data that showed how frequently each subject was chosen. All of the subjects that could potentially be eliminated were placed in a table, along with the count of students who preferred that subject.

Subject	Frequency
Industrial Arts	25
Tennis	37
Drama	83
Band	42
Debate	13
Web Design	15

Based on these results, a decision was made to cancel all of the electives in the table, except for Drama class. Unfortunately for administrators, they had to go back on their decision when students pointed out that 90% of the students in the drama class were female. The table did not clearly show the preferences of all students. It is a one-way frequency table, that could also be displayed as a bar chart.

Many frequency tables are two-way frequency tables that divide up the count, often by gender.

The table below shows the results of a clinical trial of M-54, a new medication that researchers developed to treat insomnia.

	Improved	Not Improved	Total
M-54	35	15	50
Placebo	29	71	100
Total	64	86	150

The totals for each row and each column are called the **marginal frequencies**, and the other entries (in the middle of the table) are the **joint frequencies**.

At first glance, it may look as if a placebo, which is a pill that does not contain any medication, works nearly as well for insomnia as M-54. However, if you look at the marginal frequency for each row, you can see that there were twice as many people in the placebo group as in the treatment group. In this case it would be more helpful to show the **relative frequencies**. Relative frequencies are ratios that can be calculated for the entire table, or by row, or by column. If we use only the rows or only the columns, that is called **conditional relative frequency**. For this table we would want to know whether a patient improved, given the condition that they took M-54, or given that they took a placebo. The rows show the given condition, so we divide each entry in a row by the total for that row.

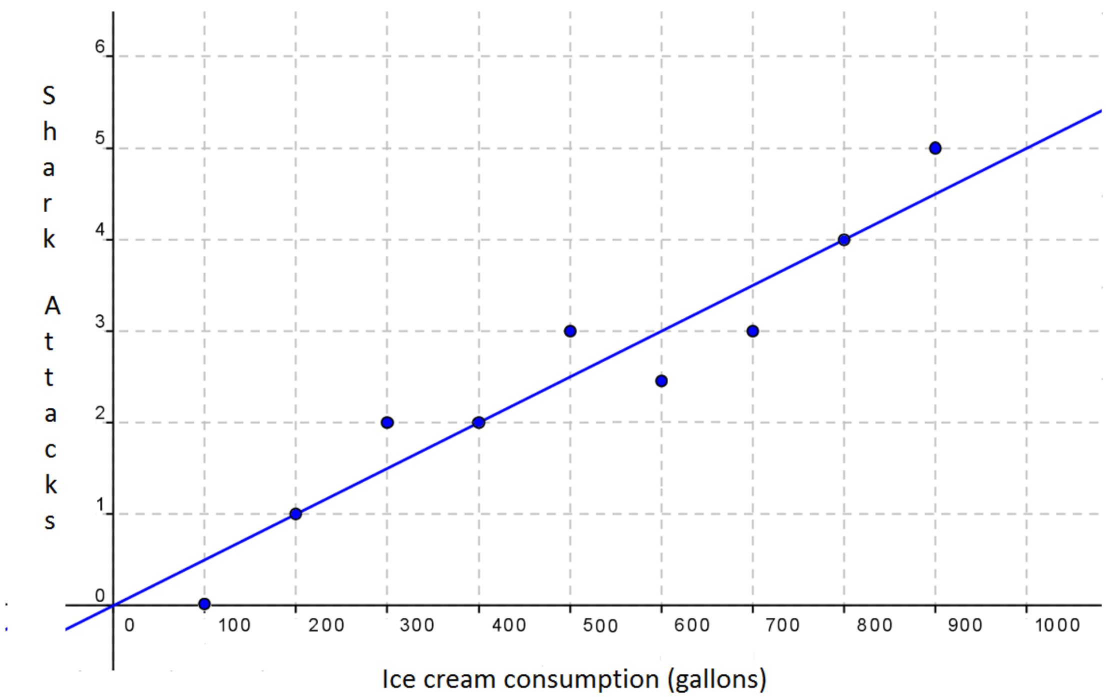
	Improved	Not Improved	Total
M-54	$\frac{35}{50} = 0.70$	$\frac{15}{50} = 0.30$	$\frac{50}{50} = 1$
Placebo	$\frac{29}{100} = 0.29$	$\frac{71}{100} = 0.71$	$\frac{50}{50} = 1$
Total	$\frac{64}{150} \approx 0.43$	$\frac{86}{150} \approx 0.57$	$\frac{150}{150} = 1$

A total of 50 people took M-54, and 70 percent of those improved. Only 29% of people who took a placebo reported improvement. Now you can see that M-54 really does look promising. Further analysis showed that the results were **statistically significant**, which means that it is not likely that random chance produced results that seemed to favor M-54.

Sometimes you are asked to determine conditional relative frequencies for a specific part of a regular frequency table. The condition is usually expressed as “given that”. For example, you could be asked to determine the conditional relative frequency that a patient actually took a placebo, given that the patient improved. Looking at the original table (the first table), we see that a total of 64 patients improved. Out of those 64 people, 29 took a placebo. 29/64 = .45, which means that if you randomly selected an improved patient there would be a 45% chance that this person had taken a placebo instead of the medicine.

Sharks and Slopes

The idea of slopes is very useful in statistics. Data, which are observed or measured values, can be recorded as dots on a graph. The position of the dots may show that there is a relationship between the two quantities, and often that relationship is roughly in the shape of a straight line (a **linear relationship**). In the graph below, ice cream consumption in a large beach resort area is graphed against the number of shark attacks. The blue line is the best "fit" for all the dots. The process of creating a line from a set of data points is called **linear regression**. The line is not just an estimate – it is carefully placed so that the dots above the line are balanced with the dots below the line. To create this balance, we have to consider the distance between each dot and the line. Sometimes a dot may be exactly on the line, so that the distance is zero. If the dot is above the line we consider the difference in position between the dot and the closest point on the line to be positive, and if the dot is below the line we record the difference as being negative. These differences are called **residuals**. When the line is perfectly placed, the residuals above the line cancel out the residuals below the line, so that the sum of all the residuals is zero.



We can use the slope of the blue line to examine the relationship between the two quantities in the graph. Look at the scale carefully to find the slope. The "rise" is 1 for every 200 units of "run". This means that on average there is one additional shark attack for every additional 200 gallons of ice cream consumed. Statistics are often used to predict what will happen to one of the quantities if the other one increases. Any prediction that goes beyond what the actual graph shows may or may not be accurate. In general, the further away you get from the actual values in the graph the less accurate your prediction is likely to be. In this case, we might predict that if ice cream consumption increases to 1200 gallons there will be 6 shark attacks.

At this point you may wonder if beachgoers should be warned not to eat ice cream. Maybe those sharks can smell traces of ice cream and they are trying to get dessert along with their meal?

Actually, this example shows one of the problems with using statistics. There is a relationship between two quantities, but it is not necessarily a cause–and–effect relationship. In this case, when the weather is nice the resort area is full of tourists. These tourists eat a lot of ice cream, and they swim in the ocean. The more people there are in the water the more likely it is that someone will be bitten by a shark. The two quantities are both related to a third quantity not shown in the graph – the temperature.

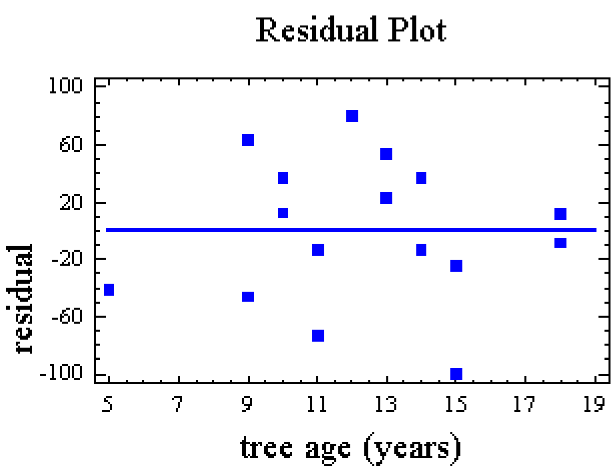
Correlation

The graph in the previous section showed more shark attacks happening as ice cream consumption increased. We say that there is a positive correlation between these two variables. To describe how close the data points are to forming a straight line, we use a **correlation coefficient**. When all of the data points are exactly on a line, the correlation coefficient has its maximum value of 1. If there is no linear relationship between the variables at all, the correlation coefficient is zero. When one variable decreases as the other increases, the correlation coefficient is negative.

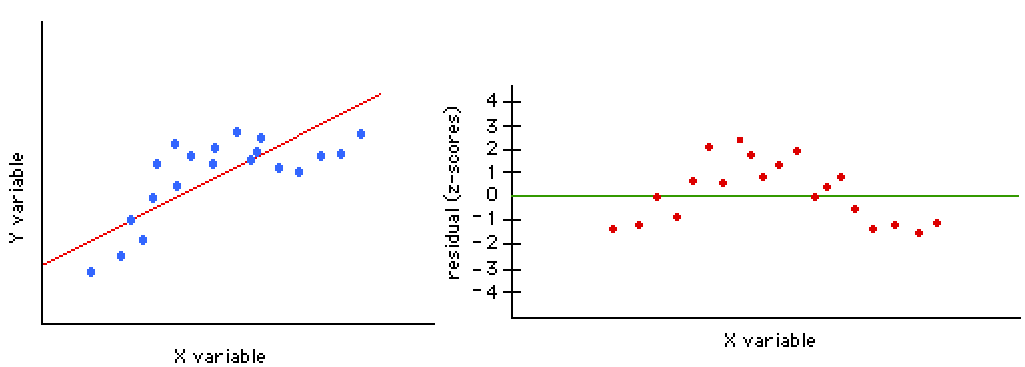
The value of the correlation coefficient is always between -1 and 1.

Residual Plots

A **residual** is the difference between the observed value and the value predicted by the line or curve that we try to fit to the data. Residuals may be positive or negative, or zero. The sum of the residuals is always zero, and so is their mean. Residuals can be plotted on a graph, just like data points. The y-axis of the graph has to extend into the negative area to accommodate all of the values of the residuals. Each residual is graphed as a y-value at the corresponding x-value for which it was determined. This results in a residual plot. An example of a residual plot is shown below:



Data are often scattered on a graph and it may not be easy to see if there is a linear relationship. When a line is a good fit for the data, some data points will appear above the line and some below, in a random pattern. The residual plot in the picture above shows this kind of random pattern. We are not looking for how close the points are to the zero line (the blue line), but only checking if the pattern seems random. If it is not, that may suggest that a function other than a linear function could be a better fit. In the image below, you can see a data set on which a linear regression has been carried out. The residuals do *not* form a random pattern:

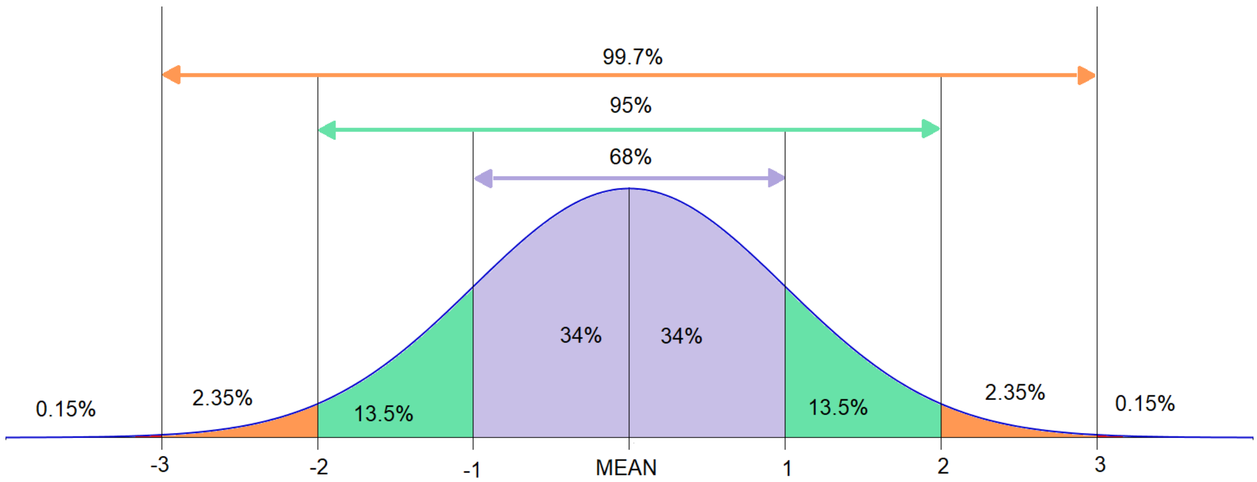


The left image shows the actual data points. The right image shows the residuals for each data point.

If you see that a line is not a good fit, you can pick a different type of function and plot the residuals again, until you get the random pattern you need.

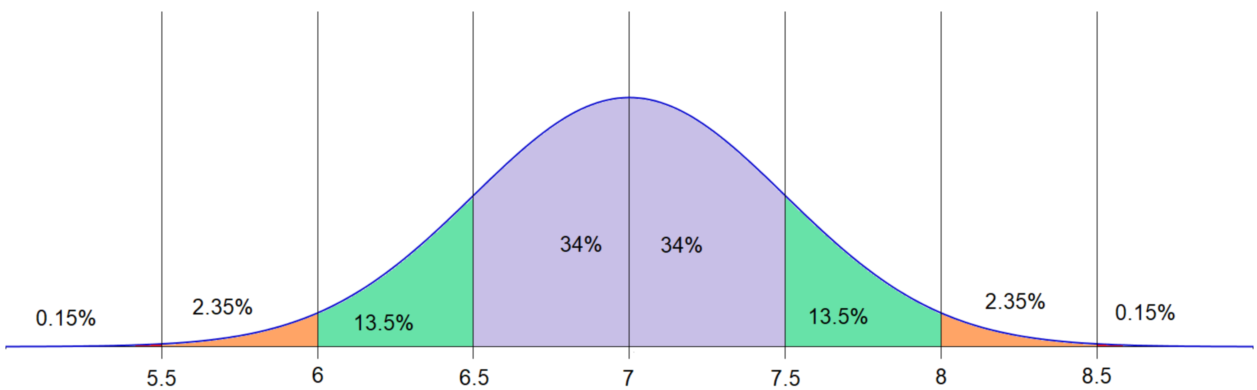
Normal Distribution

Things that vary randomly often cluster near the center of the measured values, in a very specific pattern that is called a normal distribution. For example, if you measure the height of a large number of adult men, most of your measurements will be fairly close to the average, and the further you get away from that the fewer of your measurements fall in that range. That makes sense, since most people are about average height and fewer people are very tall or exceptionally short. Most continuous biological variation, like weight, height, or student test scores, tends to show a normal distribution pattern when you graph a large number of measurements.



The area under the normal distribution curve represents 100% of the data. The **empirical rule** says that for any normal distribution, approximately 68% of the data lie within one standard deviation of the mean (between 1 below and 1 above it), about 95% of the data will be within 2 standard deviations, and by the time you get 3 standard deviations away from the mean in either direction you will have included approximately 99.7% of all of the results. This is true regardless of the actual size of the standard deviation, or the value of the mean. The actual curve never really goes to zero at the ends, but only a total of 0.3% of the data will be more than 3 standard deviations away from the mean.

For actual problems involving normal distributions, you will have to create your own scale along the bottom of the curve. For example, if samples from Ben's pear orchard indicate that the average pear weighs 7 ounces, with a standard deviation of 1/2 ounce, you can mark a scale for this normal distribution of pear weights like this:

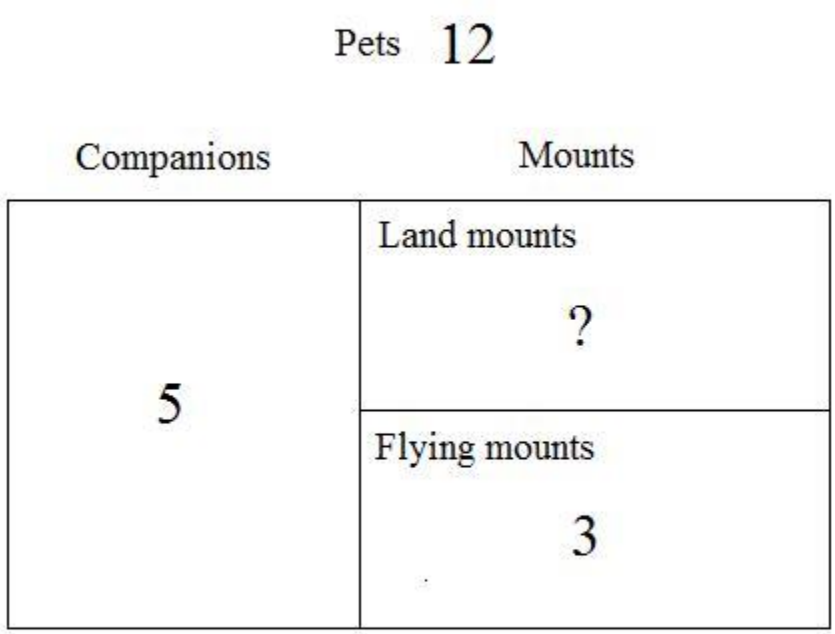


Ben would like to sell some of his pears to FancyFruits Gifts Inc., but this company will only accept pears that are 7.5 ounces or larger. What percentage of Ben's crop can be sold to FancyFruits Gifts? Well, if you look at the distribution curve you can see that 13.5% of pears will be between 7.5 and 8 ounces, 2.35% will be between 8 and 8.5 ounces, and 0.15% will be larger than 8.5 ounces. That means that based on the normal distribution we can predict that a total of 13.5 + 2.35 + 0.15 or 16% of Ben's crop will qualify. Depending on his labor costs, it may not be worth sorting his pears for sale to FancyFruits even if they pay more. To get the 16% figure faster, you can also reason that 50% of all the pears are more than 7 ounces, and then subtract the 34% that are between 7 and 7.5.

Chapter 21: Collecting Stuff (Sets)

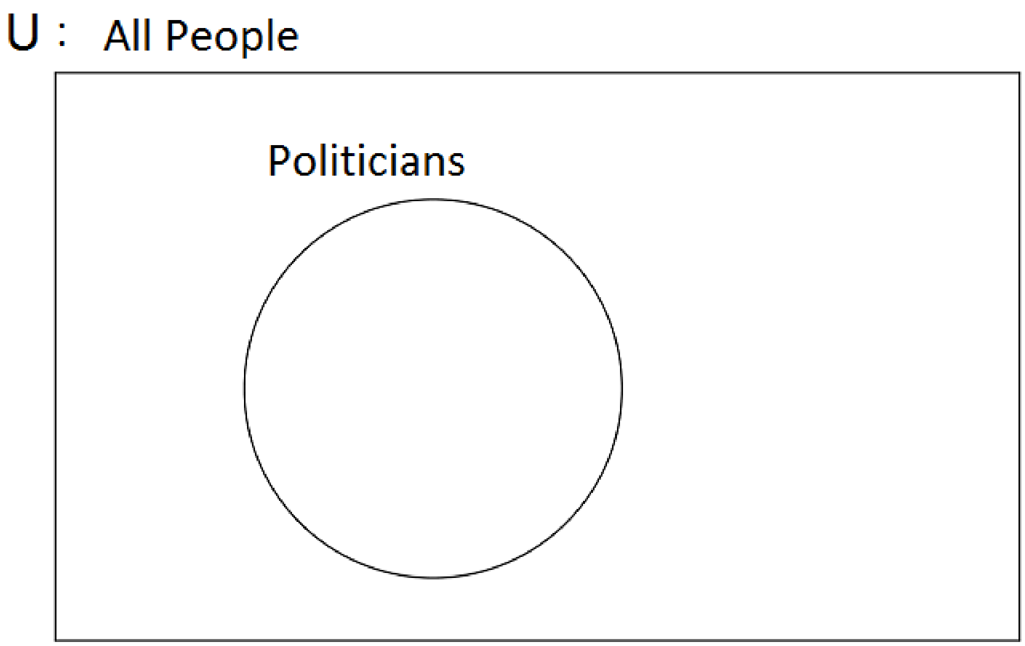
In the days before computers, people used to send each other paper e–mails called "letters". These letters had to be mailed in an envelope with a stamp on it. Because people didn't have to delete all their junk e–mail they had a lot of time on their hands, and many people would take the stamps off the envelope and collect them. This was especially interesting if you had a friend in another country, so you could divide your collection of stamps into U.S. stamps and, for example, German stamps.

These days people are more likely to have virtual collections. My favorite game, World of Warcraft, allows me to have a collection of pets. The game conveniently sorts my pets into two categories: companions, and mounts (animals I can ride on). I've further sorted my mounts into two categories: land mounts, and flying mounts. When such categories have no overlap, we can easily put them into a diagram by using simple rectangles. Why would you want to put them into a diagram? Well, a diagram helps you to solve all those pesky word problems that keep popping up on math tests. For example: "Lothar has 12 pets. 5 of those are companions, and 3 are flying mounts. How many land mounts does he have?

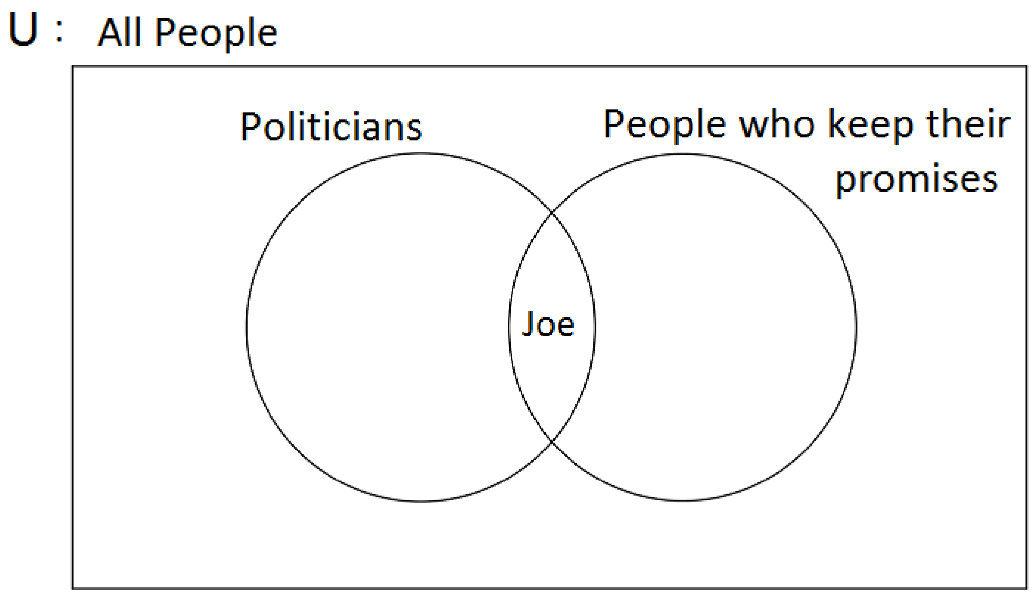


Drawing a diagram helps you see quickly that the missing number has to be 4. The more complex the problem is, the more helpful it is to use a diagram.

Sometimes things cannot be sorted into mutually exclusive categories. An item may belong in more than one category. This is more common with collections we make in our minds, which can be collections of objects, animals, people, or even ideas. Mr. John Venn (1834 – 1923) came up with a simple way of representing such collections. Venn diagrams use circles for each category, and these circles can overlap. As an example, let's create a collection of people, preferably a specific group of people we can easily make fun of, like politicians. Here is how we would represent the collection of all politicians using a Venn diagram:

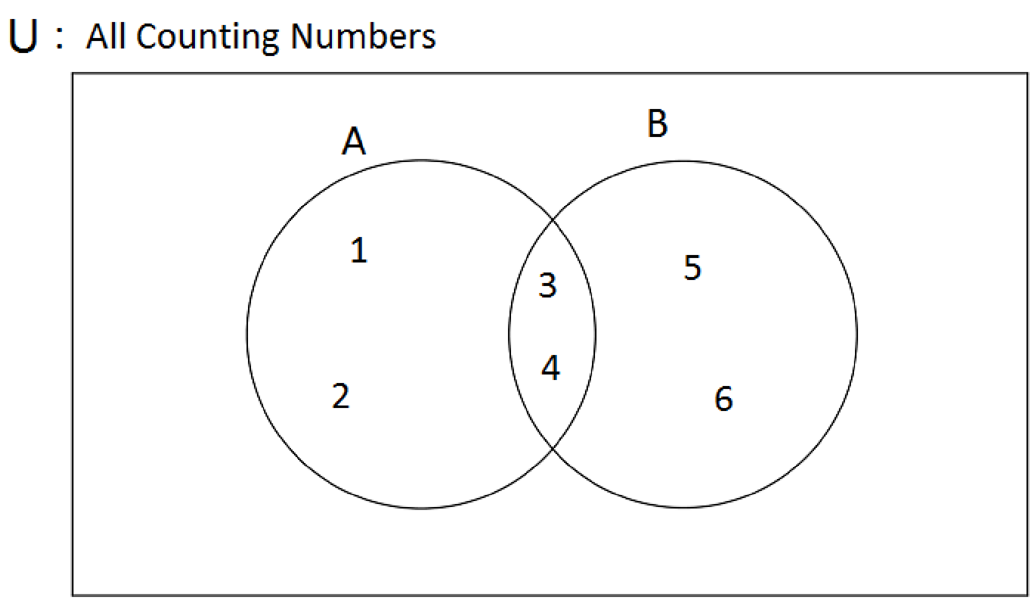


Notice that the circle is inside a box. This box represents all possible elements from which we created our specific collection, in this case all the people on Earth. The box is often labeled with a U, which stands for **Universal set**. The collection, or set, of all politicians is a **subset** of all people. Another thing to note is that the size of the circle is not related to the size of our set. We just draw the circle to be a convenient size, and imagine that it contains all the members, or **elements**, of our set. Next, let's create another set: the collection of all people who always keep their promises.



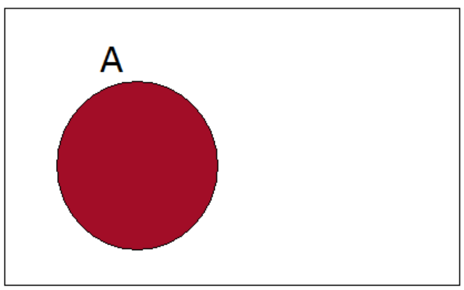
There seems to be some overlap between these two categories. This overlap is called the **intersection** of the two sets. Sometimes the intersection is empty, but in this case it looks like there is someone there. Unfortunately, Joe didn't make it in the last election.

The intersection of two sets is indicated by the sign ∩. A ∩ B would represent the overlapping regions of set A and set B. If set A = {1,2,3,4} and set B = {3,4,5,6} then A ∩ B = {3,4}:

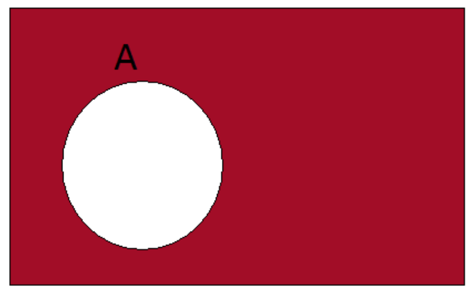


If we want all of the elements in both sets we would write A ∪ B. You can read this as "the **union** of A and B". A ∪ B = {1,2,3,4,5,6}

Sometimes you want to exclude members of a certain set. Suppose that I don't like politicians. I can decide that I will select my friends from all people who are not members of the set of politicians. If the set of politicians is called A, then all my friends would be chosen from the **complement** of set A, which may be written as A' or ~A (read that last one as *not* A).

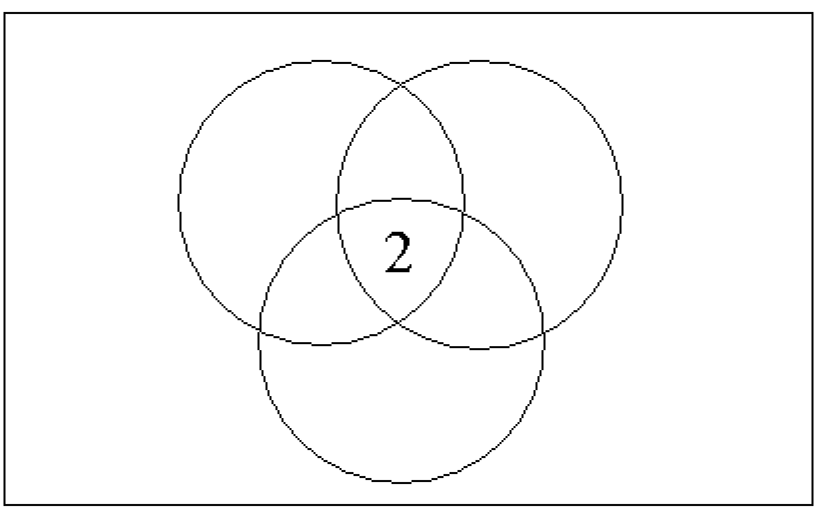


A



A'

Math tests may contain problems involving three overlapping circles, which would look like this:



When you know the number of elements that belong in a particular section, you should write that number on the diagram. In this case, there are 2 things that belong in all three categories. Once you fill in all the information given in the problem, you should be able to find the missing number by looking at your diagram and adding things up carefully.

While other people collect stamps or virtual animals, the most natural things for a mathematician to collect are numbers. There is only one problem with such a collection – it may be infinitely large. To learn more about this, check out the next section.

My Collection is Bigger than Yours

Suppose I make a collection of all the counting numbers: {1, 2, 3, 4, ...}. How big is this collection? Well, it is infinitely large, so now I have the biggest collection in the world, right? Actually when you are dealing with something that big, things are not always what they seem. Infinity is an infinitely fascinating subject.

To help students understand infinity, people often use a story about Hotel Infinity. There are many versions of this story online, but basically it involves a hotel with infinitely many rooms. Eventually the hotel seems to be full, but then another guest arrives. The clever manager finds a room for this guest by asking everyone in the hotel to move to the next numbered room. A guest in room 10 would move to room 11, and so on. This leaves room 1 empty for the new guest. Then an infinitely long bus pulls up with infinitely many passengers who also need a room in the hotel. Again the manager solves this problem. He asks each guest to move to a room with a number that is double that of their current room. All of the odd numbered rooms are now empty, and the hotel can accommodate all of the additional guests. Infinity is indeed very large, and it can accommodate additional infinities within it.

Comparing the set of all odd numbers, {1, 3, 5, 7, ...} to the set of all of the counting numbers, {1, 2, 3, 4, ...}, would you say that the second set is larger? Can you prove your statement? Georg Cantor single–handedly tackled the mind–boggling subject of infinity, even though his own mind was in such rough shape that he periodically had to rest it in the local mental hospital. Now that is true dedication to mathematics. Cantor considered the relative sizes of infinite sets by matching the elements of one set to the elements of another, one by one. In our example, he would match 1 with 1, 2 with 3, 3 with 5, etc. Even though the set of odd numbers may seem smaller, there is always a new odd number that you can find to match with the next counting number. The two sets are the same size.

Cantor considered many infinite sets, trying to find one that was even larger. Eventually he succeeded. Consider the set of all possible combinations of the counting numbers. It would look sort of like this: {(1, 2, 3), (1, 2) (1, 3), (2, 3), (1, 2, 3, 4), (1, 4), (2, 3, 4) …, (7, 18, 101, 1300, 2008, 9085), ...}. Each part of this set is a set in itself, so now we have a collection of sets. This collection certainly seems extremely large, and Cantor proved that it is in fact larger than the infinitely large set of counting numbers. To accomplish this, he matched each separate set with a counting number. In the example above, you would match (1, 2, 3) with 1, (1, 2) with 2, (1, 3) with 3, (2, 3) with 4, (1, 2, 3, 4) with 5, and so on. Notice that some sets have to be matched with numbers that are not in them. (1, 2, 3) can be matched with the number 1, but soon we run out of numbers and (2, 3) had to be matched with 4. That doesn't seem like a problem because we have lots of counting numbers available, but Cantor discovered something amazing.

Consider all of the counting numbers that were matched with some set that they were not a part of. That collection of numbers is also a set. Let's call it set A. Set A is part of our total collection of sets, so we should match it with a counting number. But what number shall we pick? If we take a counting number that is already in set A, …. hmm that wouldn't work because those are all numbers matched with combinations that don't contain them. Okay, then we have to pick a counting number that is not in set A. Unfortunately, as soon as we do that number is matched with a set, set A, that doesn't contain that number. By definition, that number now immediately becomes part of set A and we can't pick it.

This odd proof is known as Cantor's Theorem, and it shows that some infinities are bigger than others. If you find it confusing, well you're not alone.

Leopold Kronecker, Cantor's former teacher, disliked him and used his influence to keep Cantor's work from being accepted by the mathematical community. Although Kronecker made some important contributions to mathematics himself, history remembers him mainly as the fool who failed to recognize the great Cantor's brilliant work.

Chapter 21 Quiz

Question 1

Mrs. Pritt's sixth grade classroom library has 52 books. Books are either hardcover or softcover, and either fiction or non–fiction. Forty of the books are hardcover. Two of the softcover books are non–fiction. There are 11 fiction books in the library. How many books are hardcover non–fiction books?

The categories in this problem have no overlap. Solve the problem using a simple diagram.

Question 2

Cornerstone homeschool group organizes clubs for its members. The math club has 15 members and the chess club has 12 members, with some students attending both clubs. If 8 students belong only to the chess club, how many belong only to the math club? Solve this problem using a Venn diagram.

Question 3

Mr. Rand's kindergarten class is learning about pets. Of the 26 students in the class, 14 have dogs, 10 have cats, and 5 have fish. Four children have dogs and cats, 3 have dogs and fish, and one has a cat and fish. If no one has all three kinds of pets, how many students have none of these pets?

Question 4

Here is a problem that requires you to be a little more creative in drawing your circles. It uses the fact that at a large family gathering, people can be placed into more than one category in terms of family relationships.

At one family reunion, every niece was a cousin. Half of all aunts were cousins. Half of all cousins were nieces. There were 50 aunts and 30 nieces. No aunt was a niece. How many cousins were neither nieces nor aunts?

Useful Skills

1. Know how to find the perimeter and area of a rectangle.
2. Create an algebraic expression from written descriptions of a problem.
3. Create a function to represent a particular situation.
4. Find the domain and range of a function.
5. Solve equations with one variable.
6. Solve inequalities with one variable.
7. Create and solve proportions.
8. Rearrange equations with more than one variable.
9. Create and solve systems of equations using algebra and/or graphs.
10. Find the slope of a line, and the slope of a line parallel or perpendicular to it.
11. Understand real–life meanings of the slope of a graph.
12. Graph a linear equation.
13. Create an equation for a line from a graph or from two given points.
14. Determine the x and y intercepts of a line.
15. Graph a linear inequality.
16. Work with exponents.
17. Work with square roots.
18. Recognize graphs of quadratic equations.
19. Understand the meaning of the x–intercepts of the graph of a quadratic equation.
20. Know how to factor: Simple factoring, ac method, difference of two squares.
21. Know how to complete the square, and use the quadratic formula.
22. Create a boxplot to represent a set of data.

PSAT Practice

The PSAT is an important test for several reasons. Before you take it you should be very familiar with the kinds of problems you'll be expected to solve. Required skills are adjusted periodically based on the current algebra 1 curriculum, and you should practice now as well as shortly before you take the test.

Search for "College Board PSAT practice" to find sample questions. There may be geometry questions, which you can skip until you complete at least part of a geometry course.

Final Test

The following slightly modified test was designed to measure basic algebra skills.

Question 1

Evaluate the expression $\frac{3a+2b}{2}$ when a = −3 and b = −4.

a. $-\frac{1}{2}$

b. $-\frac{17}{2}$

c. $\frac{1}{2}$

d. $\frac{17}{2}$

Question 2

Simplify: $3 + 5 \cdot 6 - 4$

a. 17

b. 29

c. 30

d. 16

Question 3

Simplify: $6 - 2 \cdot 2 + 2^5$

a. 40

b. 12

c. 18

d. 34

Question 4

Evaluate: $\frac{3x-y}{6z-x}$ if x = 2, y = 8 and z = −2

a. $-\frac{1}{7}$

b. $\frac{1}{7}$

c. $\frac{1}{5}$

d. $-\frac{1}{5}$

Question 5

Simplify: $\frac{14-30}{2(-4)}$

a. $-\frac{11}{2}$

b. $\frac{11}{2}$

c. 2

d. −2

Question 6

Use the distributive property to simplify: −3(x − 10) + x

a. −2x − 30

b. −4x + 30

c. $-2x + 30$

d. $-4x - 30$

Question 7

Simplify: $8y - 2 - 3(y - 4)$

a. $11y - 6$

b. $5y - 14$

c. $5y - 6$

d. $5y + 10$

Question 8

Write the fraction in lowest terms: $\frac{36a^3bc^2}{24ab^4c^2}$

a. $\frac{2a^2}{3b^3}$

b. $\frac{2b^3}{3a^2}$

c. $\frac{2b^2}{3a^3}$

d. $\frac{3a^2}{2b^3}$

Question 9

Solve for x: $3(x + 1) = -6$

a. $\frac{7}{3}$

b. -2

c. -3

d. 1

Question 10

Add: $2a + 3b + 5a - 7b$

a. $7a + 4b$

b. $7a - 10b$

c. $7a - 4b$

d. $3ab$

Question 11

Subtract the polynomials: $(9x^2 - 4x + 11) - (3x^2 - 2x + 2)$

a. $6x^2 - 2x + 13$

b. $6x^2 - 6x + 13$

c. $6x^2 - 6x + 9$

d. $6x^2 - 2x + 9$

Question 12

$(x + 2)(x^2 - 2x + 4)=$

a. $x^3 + 8$

b. $x^3 - 4x^2 + 8x + 8$

c. $x^3 + 8x + 8$

d. $x^3 + 4x^2 - 8x + 8$

Question 13

The difference of twice a number and six is four times the number. Find an equation to solve for the number.

a. $2x - 6 = 4x$

b. $2x - 6 = 4$

c. $2x + 6 = 4x$

d. $2x - 6 = x + 4$

Question 14

Expand: $(2x - 3)^2$

a. $4x^2 + 9$

b. $4x^2 - 12x + 9$

c. $2x^2 - 12x + 9$

d. $4x^2 - 9$

Question 15

If $x > 0$, which of these numbers is the smallest?

a. $-1x$

b. $-\frac{3x}{4}$

c. $-\frac{2x}{3}$

d. $-\frac{3x}{2}$

Question 16

Which of the following is the largest?

a. $|5 - 2| + |2 - 5|$

b. $|2 - 5|$

c. $|5 - 2|$

d. $|-2 - 5|$

Question 17

Solve: $3(x - 5) \leq x - 8$

a. $x \leq \frac{2}{7}$

b. $x \leq \frac{7}{2}$

c. $x \leq -1$

d. $x \leq -1$

Question 18

A flower bed is in the shape of a triangle with one side twice the length of the shortest side and the third side 15 feet longer than the shortest side. If the perimeter is 100 feet and x represents the length of the shortest side, find an equation to solve for the lengths of the three sides.

a. $x + 2x = x + 115$

b. $x + 2x + x + 15 = 100$

c. $x + 15 + 2x = 100$

d. $x + 15 = 2x$

Question 19

If John has 10 more raffle tickets than Mary and you choose to represent John's number of raffle tickets as x, how should you represent Mary's number of raffle tickets in terms of x?

a. $x + 10$

b. $10 - x$

c. $x - 10$

d. $10 \cdot x$

Question 20

Multiply: $2x(3x^2 - 5x - 3)$

a. $6x^3 - 5x^2 - 6x$

b. $6x^3 - 10x^2 - 6x$

c. $6x^3 - 5x - 3$

d. $6x^3 - 10x^2 - 3x$

Question 21

Divide: $\frac{14m^2 - 28m^8 + 7m}{7m}$

a. $2m - 28m^8 + 7m$

b. $2m - 4m^7$

c. $2m - 4m^7 + 1$

d. $2m^2 - 4m^8 + m$

Question 22

Factor completely: $12x^4 - 20x^3 + 4x^2$

One factor is:

a. $3x^2 - 5x + 1$

b. $x - 1$

c. $4x^4$

d. $3x + 1$

Question 23

Factor completely: $x^2 - 12x + 36$

One factor is:

a. $6x$

b. $x + 3$

c. $x - 6$

d. $x - 12$

Question 24

Factor completely: $7x^2 + 14x - 21$

One factor is:

a. $x + 3$

b. $x + 1$

c. $7x$

d. $x - 3$

Question 25

Solve: $x^2 - 3x - 10 = 0$

One solution is:

a. $x = 1$

b. $x = -2$

c. $x = 10$

d. $x = 2$

Question 26

Solve: $2x^2 - 5x = 0$

The solutions are:

a. $x = 0$

b. $x = 0, x = 5$

c. $x = 0, x = \frac{5}{2}$

d. $x = 0, x = -5$

Question 27

Simplify and reduce: $\frac{3x^2-12}{9x+18}$

a. $\frac{x-2}{6}$

b. $3x - \frac{2}{3}$

c. $\frac{x+2}{3}$

d. $\frac{x-2}{3}$

Question 28

Given the equation $-2x + 3y = 12$, find the missing value in the ordered pair (−3, __)

a. −2

b. 6

c. 2

d. −6

Question 29

Consider the graph of the equation $y = -4x - 8$. What are the coordinates of the x–intercept of this graph?

a. (−2,0)

b. (2,0)

c. (0,2)

d. (0,−2)

Question 30

What is the slope of the line that is perpendicular to $y = 3x + 6$?

a. $-\frac{1}{3}$

b. $\frac{1}{3}$

c. 3

d. −3

Question 31

Solve and simplify if possible: $\frac{x^2}{x-3} - \frac{9}{x-3}$

a. $\frac{x}{-3} - \frac{3}{x}$

b. −1

c. x − 3

d. x + 3

Question 32

Solve the following system of equations for the y–value:

x + 2y = 7

2x + 2y = 13

a. y = 6

b. $y = \frac{1}{2}$

c. y = 5

d. $y = \frac{13}{4}$

Question 33

$\frac{5}{6a} - \frac{2}{3a^2} =$

a. $\frac{3}{6a^2}$

b. $\frac{3}{3a}$

c. $\frac{1}{6a^2}$

d. $\frac{5a-4}{6a^2}$

Question 34

$\frac{9b^2-3b}{3b} =$

a. 9b

b. $3b - 1$

c. $9b^2 - 1$

d. $b - 1$

Question 35

Hieronymus Huck High School has 5 more than three times as many female as male teachers. If "x" represents the number of male teachers write an expression that would represent the total number of female teachers in terms of x.

a. $3(x + 5)$

b. $x + 5$

c. 15x

d. $3x + 5$

Question 36

Which of the following is not a correct statement:

a. $x^{-4} = \frac{1}{x^4}$

b. $(x^4)^3 = x^7$

c. $4 - \frac{b^2}{25} = (2 - \frac{b}{5})(2 + \frac{b}{5})$

d. $x^5 - 3x^2 = x^2(x^3 - 3)$

Question 37

Samantha needs enough fencing to enclose a rectangular garden with a perimeter of 140 feet. If the width of her garden is to be 30 feet, write the equation that can be used to solve for the length of the garden.

a. $140 - 2x = 60$

b. $2x + 60 = 140$

c. $2x + 30 = 140$

d. $x + 30 = 140$

Question 38

Which of the following ordered pairs is NOT a solution for the equation $3x + y = 12$?

a. (6,2)

b. $(\frac{1}{2}, 10\frac{1}{2})$

c. (2,6)

d. (12,–24)

Question 39

Consider the expression $\frac{x^2+4x+4}{x^2+x-6}$ For what value(s) of x will this expression be undefined?

a. 6

b. –2

c. 2 and –3

d. –2 and 3

Question 40

$x^2 + 4x - 5$

$2x^2 + 3x - 5$

Which of the following is a factor of both expressions?

a. (x – 3)

b. (x – 5)

c. (x + 5)

d. (x – 1)

Answers

Chapter 1 Quiz

1. a: $l + s + b + j + c > 1000$

2. b: $200 - m = 10.5$ (Dollar signs are left out of the equation.)

3. b: makes complex problems easier to work with

4. $8 + 40 \div 8 - 2 \times 5 - 3 = 0$

5. $20 - (6 + 4) = 20 - 6 - 4 = 10$, or $20 - (6 + 4) = 20 - (10) = 10$

6. $35 - (7 - 3) = 35 - 7 + 3 = 31$, or $35 - (7 - 3) = 35 - 4 = 31$

7. $g + z = 34$

8. True

9. d: $a < c$

10. True. The sign means greater than *or* equal to, and 4 is equal to 4.

11. $10^4 = 10{,}000$

12. $19^1 = 19$

13. $5^2 + 5^3 = 25 + 125 = 150$

14. a) T b) F c) T d) T e) F

Chapter 1: Binary Quiz

1. 10001

2. 5

3. 10010

4. 111

Chapter 2 Quiz

1. 4 – 7 = -3

2. 6 – 12 = -6

3. −3 + 5 = 2

4. −2 – 6 = -8

5. −7 + 14 = 7

6. False: −9 < 0

7. False: |−100| = 100

8. True: |−2| > −3

9. True: −4 < −3

10. True: |−2| ≥ 2

11. False: The number line was already infinitely long.

12. −3 x 4 = -12

13. 5 x −3 = -15

14. −25 x −4 = 100

15. 9 x −10 = -90

16. −1 x −1 = 1

17. $-30 \div -6 = 5$

18. $100 \div -5 = -20$

19. $71 \div 71 = 1$

20. $\frac{-50}{25} = -2$

21. a: $p \geq 0$

22. d: $n < 0$

Chapter 3 Quiz

1. $n - n = 0$

2. $n + n - n = n$

3. $m \times 1 = m$

4. $\frac{m}{m} = 1$

5. $A = s^2$, where s represents the length of the sides. Another letter can be used instead of s.

6. $P = 4s$.

7. $V = s^3$

8. Find a formula for the surface area of a cube. $A = 6s^2$

9. **a)** 20 goats.
b) 4c goats

10. **a)** 80 millimeters (80 mm)
b) 10c millimeters

11. **a)** \$40.25
b) $C = 5.25 + 15m + 0.5w$, where C is the total cost in dollars.

12. $2n + 3$

Chapter 4 Quiz

1. $3a + 5a = 8a$. For $a = 10$: $3(10) + 5(10) = 8(10)$, which says that $30 + 50 = 80$.

2. $12b - 10b = 2b$

3. $b + b + b = 3b$

4. $ab + ab + ab = 3ab$. For $a = 3$ and $b = 4$: $12 + 12 + 12 = 36$

5. $1x - 0.5x = 0.5x$

6. $x - 0.1x$, which is $0.9x$ dollars.

7. 5b bloogs.

8. y^2 means c) $y \cdot y$

9. $2 \cdot 4a = 8a$

10. $10a \cdot -5 = -50a$

11. $7b \cdot 3 = 21b$

12. $-3 \cdot -3a = 9a$

13. $x^2 + x^2 = 2x^2$

14. $x^2 \cdot x^2 = x^4$

15. $2x^2 + x + x^2 = 3x^2 + x$

16. Mark's grandfather is 3y years old.

17. $5(8 + a) = 40 + 5a$

18. $12(f + 5) = 12f + 60$

19. $3(4 - x) = 12 - 3x$

20. $-6(y - 4) = -6y +24$

21. $a(b + c) = ab + ac$

22. $-a(b - c) = -ab + ac$

23. $12x - 22 - 10(x - 2) = 2x - 2$. You could also write the answer as $2(x - 1)$.

24. $x \cdot y = xy$. $3x \cdot 3y = 9xy$. The AREA of the new rectangle is c: 9 times as big as the area of the original rectangle.

25. False. 3x means 3 times x, not 3 + x. $3x - 3$ cannot be simplified.

26. False. $5x^2 - x^2 = 4x^2$

Chapter 5: Let's Work Together

1. 5 cars

2. 12 minutes

3. Paul's hourly rate: 2 cars per hour + Mike's hourly rate 3 cars per hour = 5 cars per hour. This means 1 car takes 12 minutes.

If you use fractions:

$\frac{1}{30} + \frac{1}{20} = \frac{2}{60} + \frac{3}{60} = \frac{5}{60} = \frac{1}{12}$. The answer is in cars per minute, so 1 car per 12 minutes.

4. Using fractions: $\frac{1}{5} + \frac{1}{20} = \frac{4}{20} + \frac{1}{20} = \frac{5}{20} = \frac{1}{4}$

Put the units back to get 1 car per 4 minutes.

Using hourly rates: The first machine is working at a rate of 1 job per 5 minutes, which is the same as 12 jobs per hour. The second machine is working at a rate of 1 job per 20 minutes, which is 3 jobs per hour. Now add the two rates together: 15 jobs per hour. This is how fast both machines can work together. Working at this rate, it takes 4 minutes to do one job.

Odds and Evens

even + even	=	even
even + odd	=	odd
odd + odd	=	even
even times even	=	even
even times odd	=	even
odd times odd	=	odd

Chapter 5 Quiz

1. $\frac{-10x}{-5x} = 2$

2. $\frac{-12b}{12} = -b$

3. $-12a \div -2 = 6a$

4. $10a \div 5 = 2a$

5. $40c \div -4 = -10c$

6. $\frac{d}{d} = 1$

7. $\frac{-4ac}{2a} = -2c$

8. $\frac{x}{2} + \frac{x}{2} = x$

9. c: 3 times as big as the original perimeter

10. $40 + 5a = 5(8 + a)$

11. $12f + 60 = 12(f + 5)$

12. $12 - 3x = 3(4 - x)$

13. $6y + 24xy = 6y(1 + 4x)$

14. $ab + ac = a(b + c)$

15. $a^2 - ac = a(a - c)$

16. $4a^2 - 4ac = 4a(a - c)$

17. $4x^2 - 4ax + x = x(4x - 4a + 1)$

18. a. 10 hours

b. 2 hours

c. $\frac{100}{m}$

19. The formula for determining speed is c: $v = d / t$

20. c: $d = v \cdot t$

21. 0.04 miles per minute

22. 0.84 miles

0.16 miles

23. Alyssa is at distance d from your house after walking m minutes. This can be expressed by the formula b: $d = 1 - .04m$

24. a: $d = .03m$

25. $3 \cdot \frac{1}{x} = \frac{3}{x}$

26. $3 \cdot \frac{2}{7} = \frac{6}{7}$

27. $3 \cdot \frac{2}{x} = \frac{6}{x}$

28. $a \cdot \frac{1}{5} = \frac{a}{5}$

29. $a \cdot \frac{1}{x} = \frac{a}{x}$

30. $a \cdot \frac{b}{x} = \frac{ab}{x}$

31. $\frac{2}{5} \cdot \frac{1}{7} = \frac{2}{35}$

32. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

33. $\frac{a}{b} \cdot \frac{a}{b} = \frac{a^2}{b^2}$

34. $\frac{b}{2a} \cdot \frac{b}{2a} = \frac{b^2}{4a^2}$

35. $\frac{2}{5} \div \frac{1}{2} = \frac{4}{5}$

36. $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$

37. $\frac{2}{3} + \frac{1}{4} = \frac{11}{12}$

38. $\frac{2}{b} + \frac{1}{4} = \frac{8+b}{4b}$

39. $\frac{2}{b} + \frac{1}{d} = \frac{2d+b}{bd}$

40. $\frac{a}{b} + \frac{1}{d} = \frac{ad+b}{bd}$

41. $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$

42. $\frac{b}{4a} + \frac{c}{a} = \frac{b+4c}{4a}$

43. $\frac{x}{y} = z \qquad y = \frac{x}{z}$

44. $\frac{x}{y} = z \qquad x = yz$

45. $\frac{a}{9p} = x \qquad a = 9px$

Chapter 6 Quiz

1. x = 4

2. x = 35

3. x = −5

4. x = 2

5. x = 5

6. x = 2

7. x = −5

8. x = −3

9. x = −27

10. x = 12

11. x = 4

12. x = 15

13. $x = 1$

14. $0.12n = 108$, $n = 900$

15. $n + 5n = 90$, $n = 15$

16. $y = 0$, or $1 = 2y$ so $y = \frac{1}{2}$

Alternatively: $y - 2y^2 = 0$

$y(1 - 2y) = 0$ so $y = 0$ or $1 - 2y = 0$

Chapter 8 Quiz

1. $r = 2s - 5$. 19 times.

2. $0.04 + 0.05 = 0.09$ cars/minute.

$\frac{0.09 \text{ cars}}{1 \text{ minute}} = \frac{1 \text{ car}}{x \text{ minutes}}$. $x = 11$ minutes [round your answer to the nearest minute]

3. $\frac{5}{100} = \frac{40}{n}$, $n = 800$

4. $9680 \times \$10 = 96{,}800$.

5. a. $x + 0.5x = 16$. $x = 10\frac{2}{3}$

b. $x + \frac{1}{5}x = 21$. $5x + x = 105$. $x = 17.5$

6. 15 and 72

7. $1 - 0.04m = 0.03m$, $m = 14.29$ minutes. $d = 0.03 \cdot 14.29$, $d = 0.43$ miles

or $d = 1 - (0.04)(14.29) = 0.43$ miles

Test 1

1. $-81 \div -9 < -81 \div 9$ False

2. p is a positive number. $2p < 3p$ True

3. n is a negative number. $2n < 3n$ False

4. Divide: $\frac{-6x^2}{3x}$ = -2x (x is not equal to zero)

5. Divide: $\frac{5x+5}{5}$ = x + 1

6. Use the distributive property: $3x\,(x - 1) = 3x^2 - 3$

7. $\frac{x+9}{5} = x - 3$ $x = 6$

8. $\frac{a}{b} \cdot \frac{1}{5} = \frac{a}{5b}$

9. $\frac{1}{a} \div \frac{1}{b} = \frac{b}{a}$

10. A snail wakes up in the morning, and decides to head back to the puddle where it had a drink the night before. The edge of the puddle is 1.27 meters away. The snail starts moving directly toward the puddle at a steady speed. After 3 minutes, it has traveled 12 inches. What is the snail's speed in inches per minute? __4__ inches /min. At this speed, how long will it take for the snail to reach the puddle? __12.5__ minutes.

[Conversion factors: 1 inch = 2.54 cm 1 meter = 100 cm]

11. $x + 2 = y$. $|x - y| + |y - x| = 0$

a. 0

12. A rectangle has a perimeter of 58 cm. The length of the rectangle is 5cm longer than the width. The width of this rectangle is __12__ cm.

13. Add: $(8a + 5b) + (3a - 2b) = 11a + 3b$

14. Subtract: $(5x + 4y) - (3x - 2y) = 2x + 6y$

15. Multiply: $x(3x + 5 - y) = 3x^2 + 5x - xy$

16. $-x + \frac{1}{5} = \frac{4}{5}x + 2$ $x = -1$

17. For all positive and negative numbers a and b, $|a - b| = |b - a|$ True

18. $2x + 2 < 10$ $x <$ ___4____

19. $4x + 7 - 5x < 2$ Write the correct expression for x: ___$x > 5$___

20. $\frac{x}{a} + \frac{3}{b} = \frac{bx + 3a}{ab}$

21. If $x/y = 4$ and $y = 3$, then $xy =$ d. 36

22. If $a - b = b - 4 = 5$, then $a =$ d. 14

23. If $6x - 1 = 3a$, then $(6x - 1)/3 =$ c. a

24. b. $x > 10$

25. Joanne drives at an average speed of 60 miles per hour for 1 hour. How long would Jim have to drive at an average speed of 30 miles per hour to cover the same distance?

d. 2 hours

26. The number x lies within 3 units of 10 on a number line. We could describe this relationship as

b. $|x - 10| \leq 3$

27. A 10 ounce bottle of shampoo costs x dollars, while a 20 ounce bottle of the same shampoo costs y dollars. If the larger bottle costs z dollars less than 2 of the 10 ounce bottles together, which of the following equations must be true?

d. $2x - y - z = 0$

28. An engineer needs to know the weight of the screws he intends to use in a project. If 7 screws and a 10 gram weight balance on a scale with 3 screws and a 30 gram weight, what is the weight of one screw?

c. 5 grams

29. A machine at a factory can produce fan blades at a rate of 120 per hour. An employee can package these blades at a rate of 3 per minute. How many employees are needed to keep up with 18 machines of this kind?

b. 12

30. Three different integers have a sum that is exactly equal to their product. The average of these three integers is also equal to their sum. If two of the integers are 0 and −15, the third integer is

[Note: the word integer means a positive or negative whole number, or zero.]

a. 15

31. If $44 \cdot 2 \cdot p = 8$, then p =

b. 11

32. If $m < n < 0$, which of the following is the largest quantity?

d. $-(m + n)$

Chapter 9 Quiz

1. $m = 4b$

$m + b = 1260$

$b = 252$

2. $x = 5y$

$3y + 6 = 5y$

$6 = 2y$

$y = 3$, so $x = 15$

3. $l = 4s$

$m = 3s$

$l + m + s = 112$

$s = 14$, $m = 42$, $l = 56$

4. $l + s = 58$

$l = 3s + 10$

$l = 46$

5. c = cows, h = chickens

$c + h = 20$

$4c + 2h = 56$

$c = 8$, $h = 12$

6. $a + c = 50$

$9.75c + 29.75a = 1327.5$

8 children times 5 times \$1.95 = \$ 78.00

42 adults times 5 times \$5.95 = \$ 1249.50

7. $h + c = 5$

$0.15h + 0.4c = 1$

1 pound concentrate + 4 pounds hay.

8. Measuring from train A's initial position: $d_A = 50h$ and $d_B = 18 - 30h$.

Measuring from train B's initial position: $d_B = 30h$ and $d_A = 18 - 50h$.

When $d_A = d_B$, $h = 0.225$ hours or 13.5 minutes (9:29 am)

At 10 minutes (9.25 am) train A is $8\frac{1}{3}$ miles from its starting point. The switch will divert it. At 13.5 minutes $d_A = 11.25$ miles. The tail end of train A is at 10.95 miles from train A's initial position; the entire train is on the other track. At 13.5 minutes train B is at 6.75 miles from its initial position, which means it will not reach the switch before train A has been diverted.

a: Yes, the trains start passing each other at 9:29 a.m. (rounded to the nearest minute).

Chapter 10 Quiz

1. (0,0) and (8,4) Slope = $\frac{1}{2}$

2. (2,1) and (4,7) Slope = 3

3. (−3,5) and (3,−7) Slope = 2

4. (−8,−8) and (−1,−1) Slope = 1

5. (−1,1) and (5,1) Slope = 0

6. (0,0) and (1,10) Slope = 10

7. (0,0) and (4,2) Slope = $\frac{1}{2}$

8. (−3,−9) and (0,0) Slope = 3

9. (1,1) and (5,5) Slope = 1

10. A line has a slope of 3. If it passes through (1,2) it must also pass through:
a: (0,−1)

11. A line passes through two points: (a,b) and (c,d). The slope is:
b: (d−b) / (c−a)

12. What is the slope of the line that passes through (1,2) and (1,10)?
e: Undefined

13. 9 shark attacks

Chapter 11 Quiz

1. (−5,6): yes
(4,−5): no

2. y = −x + 1 has a slope of −1

3. The line defined by y = −x + 1 intersects the y–axis at the point (0,1)

4. The line defined by y = −x + 1 intersects the x–axis at the point (1,0)

5. y = 2x − 1 has a slope of 2 and passes through the point (3,5).

6. m = 5. y = 5x + 1.

7. y = -2x + −3, y = -2x − 3.

8. y = 0.5 x + −5.5, y = 0.5x − 5.5.

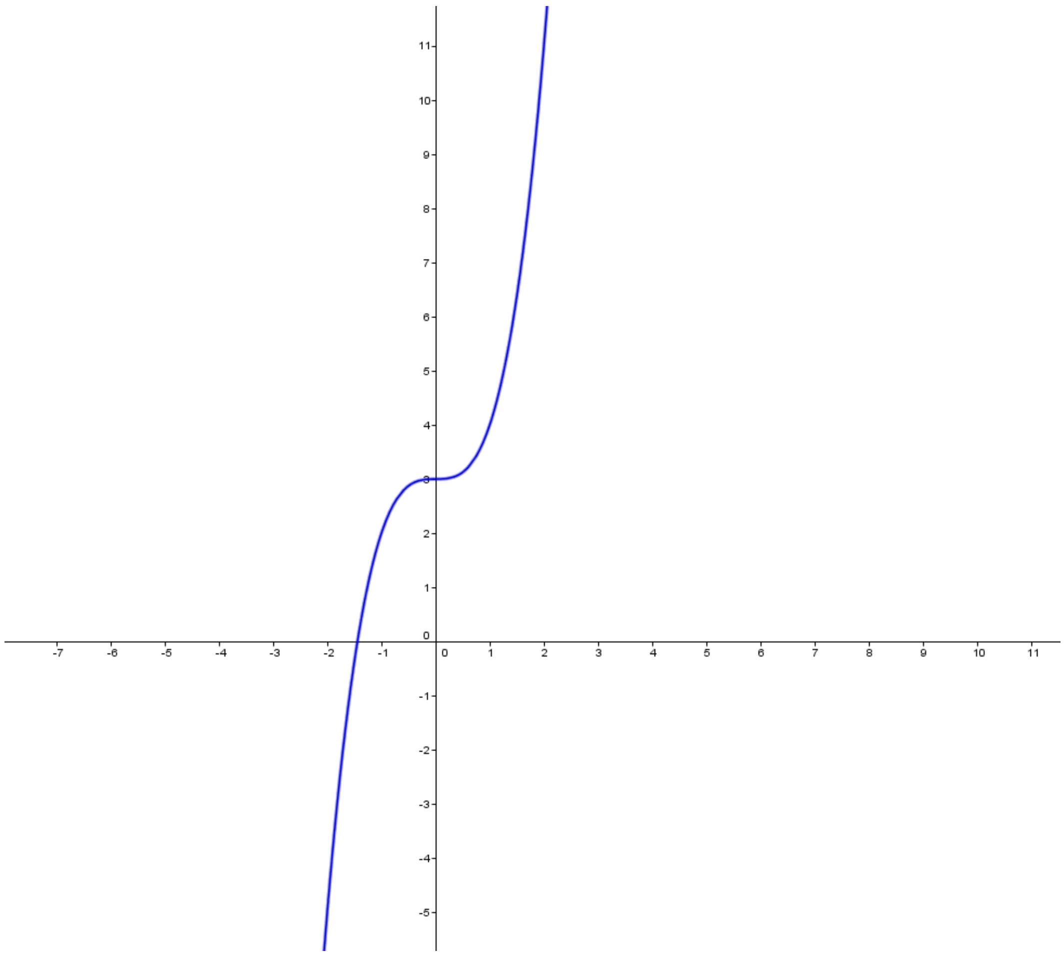
9. (2, 5)

10. y = 10x + 80

11. x = 3, y = 2
12. k = 3, y = 3x

Chapter 12 Quiz

1. Yes, each x-value has only one y-value.



2. No, this is a circle. A vertical line can touch the graph in more than one spot.

3. Yes, $y = -x + 4$ has only one y value for every x.

4. Total fee f: $f(c) = 2.99c + 98$

5. Domain: $x \geq -4$, range: $y \geq 0$

6. $f(x) = x^2 - 12$. When $x = 1$, $y = -11$ [$f(1) = -11$].

7. $f(3) = 17$.

Chapter 13 Quiz

1. $4^2 + 2^3 + 5^0 = 25$

2. $-1 \cdot 3^2 = -9$

3. $-3^2 = -9$

4. $(-3)^2 = 9$

5. $x^2 \cdot x^3 = x^5$

6. $x^3y \cdot xy^8 = x^4y^9$

7. $\frac{10^6}{10^4} = 100$

8. $\frac{x^{16}}{x^4} = x^{12}$

9. $\frac{25x^3}{5x} = 5x^2$

10. $\frac{x^{10}y^6}{x^4y^{12}} = x^6y^{-6}$ or $\frac{x^6}{y^6}$

11. $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$

12. $\left(\frac{4}{5}\right)^2 = \frac{16}{25}$

13. $\left(\frac{x}{y}\right)^2 = \frac{x^2}{y^2}$

14. $(3 \cdot 4)^2 = 144$

15. $(2b)^2 = 4b^2$

16. $(x^3)^2 = x^6$

17. $10^{-3} = 0.001$

18. $(x^2y^3)^4 = x^8y^{12}$

19. $\frac{x^{-10}}{x^5} = x^{-15}$ or $\frac{1}{x^{15}}$

20. $\frac{4}{5^{-2}} = 4 \div \frac{1}{25} = 4 \cdot \frac{25}{1} = 100$

Chapter 14 Quiz

1. **c:** 5 and −5

2. $\sqrt{\frac{25}{9}} = \frac{5}{3}$ or $1\frac{2}{3}$

3. $\sqrt{\frac{1}{4}} = \frac{1}{2}$

4. $\sqrt[3]{27} = 3$

5. $\sqrt[3]{-125} = -5$

6. $\sqrt{4 \cdot 25} = 10$

7. Simplify: $\sqrt{8} = 2\sqrt{2}$

8. Simplify: $\sqrt{75}$ = $5\sqrt{3}$

Chapter 16 Quiz

1. 5 cm

2. Yes

3. $\sqrt{2} \approx 1.414$ inches

4. $\sqrt{3} \approx 1.732$ inches

Chapter 17 Quiz

1. $x(x^2 + 10x + 16) = 0$, $x(x + 2)(x + 8) = 0$, $x = 0$ or $x = -2$ or $x = -8$.

2. $x - 3$

3. $x^2 + 21x + 20 = (x + 1)(x + 20)$: 1 and 20, a and t
$20 + 21x + x^2 = (x + 20)(x + 1)$: 20 and 1, t and a
$x^2 + 14x + 33 = (x + 3)(x + 11)$: 3 and 11, c and k
$160 + 28x + x^2 = (x + 20)(x + 8)$: 20 and 8, t and h
$x^2 + 10x + 25 = (x + 5)(x + 5)$: 5 and 5, e and e
$70 + 19x + x^2 = (x + 14)(x + 5)$: 14 and 5, n and e
$x^2 + 38x + 325 = (x + 13)(x + 25)$: m and y
$x^2 + 21x + 20 = (x + 1)(x + 20)$: a and t
$x^2 + 21x + 90 = (x + 6)(x + 15)$: f and o
$378 + 39x + x^2 = (x + 21)(x + 18)$: u and r
$x^2 + 14x + 13 = (x + 1)(x + 13)$: a and m

The message reads: Attack the enemy at four a.m.

You set your alarm for 3 a.m., but find that you are unable to sleep. At 3:55 a.m. you cautiously venture outside, armed with a baseball bat and a kitchen knife. You see several of your neighbors doing the same thing, so you join up with them and head for the local elementary school where the enemy soldiers are stationed. There are two guards at the entrance, but they are sound asleep, and don't wake up as your group moves carefully past them. Inside the building there are more soldiers, but they are also sleeping. Someone pokes a soldier with a hunting rifle and he wakes up, but he is so groggy he can't even get up. Looking around, you spot half-empty boxes of girl scout cookies. They seem to be everywhere. "No, don't!" you yell at your chubby neighbor Sam, as he has picked up a box and is about to bite into a cookie. The girl scouts must have prepared a special batch of cookies, and the enemy soldiers are now easy to round up. The battle has been won, and America is saved thanks to the girl scouts and the amazing math skills of ordinary citizens.

4. $x = -3, x = 4$

5. $(-2, -1)$ and $(2, 7)$

Chapter 19 Quiz

1. $\frac{1}{4}$

2. 720 different ways

3. 15 games

4. 1/4

5. 1/2

6. 1/2

7. $1 - \frac{1}{8} = \frac{7}{8}$

8. 2/3. The red side that you see is either the red side of the two-color card, the front side of the red card, or the back side of the red card. Two out of those three choices are the ones the question is asking about.

Chapter 21 Quiz

1. 39 books

2. 11 students

3. 5 students

4. 5 cousins

Final Test

1 b. $-\frac{17}{2}$

2 b. 29

3 d. 34

4 b. $\frac{1}{7}$

5 c. 2

6 c. $-2x + 30$

7 d. $5y + 10$

8 d. $\frac{3a^2}{2b^3}$

9 c. -3

10 c. $7a - 4b$

11 d. $6x^2 - 2x + 9$

12 a. $x^3 + 8$

13 a. $2x - 6 = 4x$

14 b. $4x^2 - 12x + 9$

15 d. $-\frac{3x}{2}$

16 d. $|-2 - 5|$

17 b. $x \leq \frac{7}{2}$

18 b. $x + 2x + x + 15 = 100$

19 c. $x - 10$

20 b. $6x^3 - 10x^2 - 6x$

21 c. $2m - 4m^7 + 1$

22 a. $3x^2 - 5x + 1$

23 c. $x - 6$

24 a. $x + 3$

25 b. $x = -2$

26 c. $x = 0$, $x = \frac{5}{2}$

27 d. $\frac{x-2}{3}$

28 c. 2

29 a. $(-2,0)$

30 a. $-\frac{1}{3}$

31 d. $x + 3$

32 b. $y = \frac{1}{2}$

33 d. $\frac{5a-4}{6a^2}$

34 b. $3b - 1$

35 $3x + 5$

36 b. $(x^4)^3 = x^7$

37 b. $2x + 60 = 140$

38 a. (6,2)

39 c. 2 and −3

40 d. (x − 1)